Math 220 – Section 10.3 Solutions

1. $f(x) = x^3 + \sin x$ is **odd** because
$$f(-x) = (-x)^3 + \sin(-x) = -x^3 - \sin x = -(x^3 + \sin x) = -f(x)$$

3. $f(x) = (1 - x^2)^{-1/2}$ is **even** because
$$f(-x) = (1 - (-x)^2)^{-1/2} = (1 - x^2)^{-1/2} = f(x)$$

5. $f(x) = e^{-x} \cos 3x$ is **neither** odd nor even because
$$f(-x) = e^{-(-x)} \cos 3(-x) = e^x \cos x$$
is neither $-f(x)$ nor $f(x)$.

9. Compute the Fourier Series for
$$f(x) = x, \quad -\pi < x < \pi$$
First, we note that $f(x)$ is odd so $a_n = 0$ for all n. Then,
$$\begin{aligned}
b_n &= \frac{1}{T} \int_{-T}^{T} f(x) \sin\left(\frac{n\pi x}{T}\right)\, dx \\
&= \frac{2}{\pi} \int_0^{\pi} x \sin(nx)\, dx \\
&= \frac{2}{\pi} \left[\frac{\sin(nx) - nx\cos(nx)}{n^2}\right]_0^{\pi} \\
&= \frac{2}{n^2\pi} \left[(\sin(n\pi) - n\pi\cos(n\pi)) - (\sin 0 - n(0)\cos 0)\right] \\
&= \frac{2}{n^2\pi} \left[(0 - n\pi(-1)^n) - (0 - 0)\right] \\
&= \frac{2}{n^2\pi} \left[-n\pi(-1)^n\right] \\
&= \frac{2(-1)^{n+1}}{n}
\end{aligned}$$
The Fourier Series for $f(x)$ is then
$$\boxed{\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx)}$$
The first few terms of the series are
$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx) &= \frac{2(-1)^{1+1}}{1} \sin x + \frac{2(-1)^{2+1}}{2} \sin 2x + \frac{2(-1)^{3+1}}{3} \sin 3x + \dots \\
&= \frac{2}{1} \sin x - \frac{2}{2} \sin 2x + \frac{2}{3} \sin 3x - \frac{2}{4} \sin 4x + \dots \\
&= 2\left(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \dots\right)
\end{aligned}$$

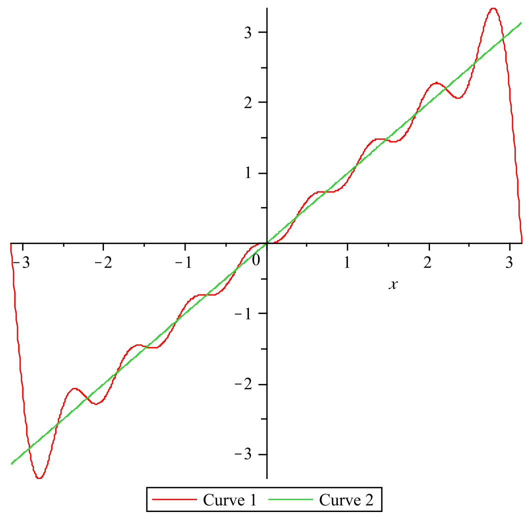


Figure 1: First 8 terms of the Fourier Series for $f(x) = x$ on $[-\pi, \pi]$.

11. Compute the Fourier Series for

$$f(x) = \begin{cases} 1, & -2 < x < 0 \\ x, & 0 < x < 2 \end{cases}$$

First, we note that $f(x)$ is neither even nor odd. Therefore, the a_n and b_n will generally be nonzero.

Let's compute a_0:

$$\begin{aligned} a_0 &= \frac{1}{2}\int_{-2}^{2} f(x)\,dx \\ &= \frac{1}{2}\int_{-2}^{0} dx + \frac{1}{2}\int_{0}^{2} x\,dx \\ &= \frac{1}{2}[x]_{-2}^{0} + \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 \\ &= \frac{1}{2}[0-(-2)] + \frac{1}{4}[2^2 - 0^2] \\ &= 2 \end{aligned}$$

Then compute the rest of the a_n:

$$\begin{aligned} a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)\,dx \\ &= \frac{1}{2}\int_{-2}^{0} \cos\left(\frac{n\pi x}{2}\right)\,dx + \frac{1}{2}\int_{0}^{2} x\cos\left(\frac{n\pi x}{2}\right)\,dx \\ &= \frac{1}{2}\left[\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right]_{-2}^{0} + \frac{1}{2}\left[\frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right) + \frac{2}{n\pi}x\sin\left(\frac{n\pi x}{2}\right)\right]_0^2 \\ &= \frac{1}{n\pi}\left[\sin 0 - \sin(-n\pi)\right] + \frac{1}{2}\left[\frac{4}{n^2\pi^2}\cos(n\pi) + \frac{2}{n\pi}(2)\sin(n\pi) - \frac{4}{n^2\pi^2}\right] \\ &= 0 + \frac{1}{2}\left[\frac{4}{n^2\pi^2}(-1)^n - \frac{4}{n^2\pi^2}\right] \\ &= \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right] \end{aligned}$$

The b_n are computed as follows:

$$\begin{aligned} b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)\,dx \\ &= \frac{1}{2}\int_{-2}^{0} \sin\left(\frac{n\pi x}{2}\right)\,dx + \frac{1}{2}\int_{0}^{2} x\sin\left(\frac{n\pi x}{2}\right)\,dx \\ &= \frac{1}{n\pi}[(-1)^n - 1] - \frac{2}{n\pi}(-1)^n \\ &= \frac{1}{n\pi}[(-1)^{n+1} - 1] \end{aligned}$$

The Fourier Series for $f(x)$ is then

$$\boxed{1 + \sum_{n=1}^{\infty}\left\{\frac{2}{n^2\pi^2}[(-1)^n - 1]\cos\left(\frac{n\pi x}{2}\right) + \frac{1}{n\pi}[(-1)^{n+1} - 1]\sin\left(\frac{n\pi x}{2}\right)\right\}}$$

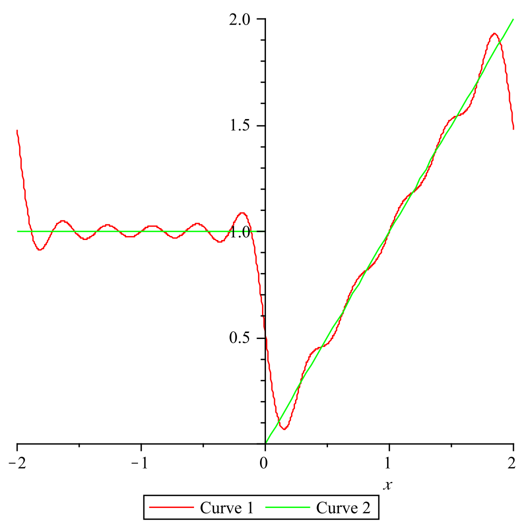
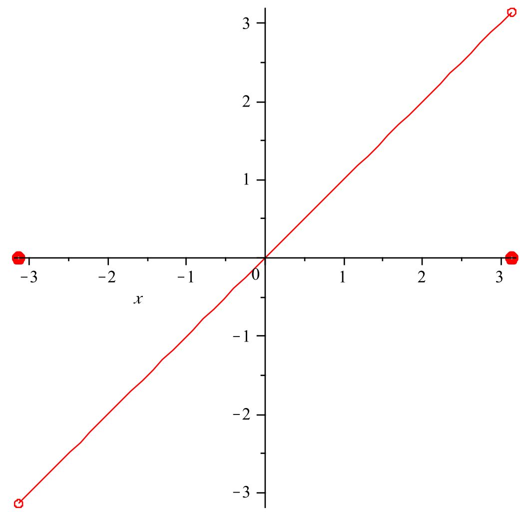


Figure 2: First 10 terms of the Fourier Series for $f(x)$ in Problem 11 on $[-2, 2]$.

17. The function to which the Fourier Series in Problem 9 converges is:

$$f(x) = \begin{cases} x, & -\pi < x < \pi \\ 0, & x = \pm\pi \end{cases}$$



19. The function to which the Fourier Series in Problem 11 converges is:

$$f(x) = \begin{cases} 1, & -2 < x < 0 \\ x, & 0 < x < 2 \\ \frac{1}{2}, & x = 0 \\ \frac{3}{2}, & x = \pm 2 \end{cases}$$