# Math 220 – Section 10.4 Solutions

1. For the function $f(x) = x^2$ on $0 < x < \pi$, we have:

   (a) The $\pi$-periodic extension of $f$ is:

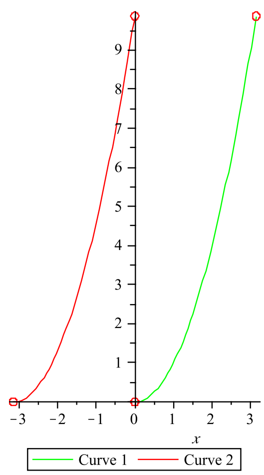
   $$\bar{f}(x) = \begin{cases} x^2, & 0 < x < \pi \\ (x+\pi)^2, & -\pi < x < 0 \end{cases}$$

   (b) The odd $2\pi$-periodic extension of $f$ is:

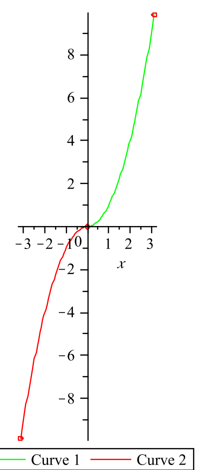
   $$f_o(x) = \begin{cases} x^2, & 0 < x < \pi \\ -x^2, & -\pi < x < 0 \end{cases}$$

   (c) The even $2\pi$-periodic extension of $f$ is:

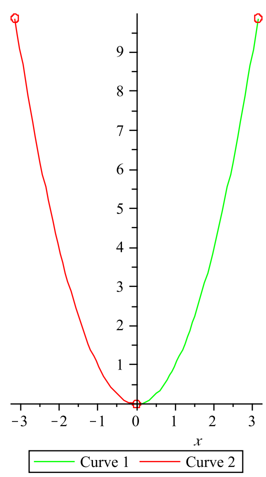
   $$f_e(x) = \begin{cases} x^2, & 0 < x < \pi \\ x^2, & -\pi < x < 0 \end{cases}$$




(a)




(b)




(c)

3. For the given function, we have:

   (a) The $\pi$-periodic extension of $f$ is:

$$\bar{f}(x) = \begin{cases} 0, & 0 < x < \pi/2 \\ 1, & \pi/2 < x < \pi \\ 0, & -\pi < x < -\pi/2 \\ 1, & -\pi/2 < x < 0 \end{cases}$$

   (b) The odd $2\pi$-periodic extension of $f$ is:

$$f_o(x) = \begin{cases} 0, & 0 < x < \pi/2 \\ 1, & \pi/2 < x < \pi \\ 0, & -\pi/2 < x < 0 \\ -1, & -\pi < x < -\pi/2 \end{cases}$$

   (c) The even $2\pi$-periodic extension of $f$ is:

$$f_e(x) = \begin{cases} 0, & 0 < x < \pi/2 \\ 1, & \pi/2 < x < \pi \\ 0, & -\pi/2 < x < 0 \\ 1, & -\pi < x < -\pi/2 \end{cases}$$

5. For the function $f(x) = -1$ on $0 < x < 1$ we have:

$$\begin{aligned} b_n &= \frac{2}{T}\int_0^T f(x)\sin\left(\frac{n\pi x}{T}\right)\,dx \\ &= \frac{2}{1}\int_0^1 (-1)\sin\left(\frac{n\pi x}{1}\right)\,dx \\ &= -2\int_0^2 \sin(n\pi x)\,dx \\ &= -2\left[-\frac{1}{n\pi}\cos(n\pi x)\right]_0^1 \\ &= \frac{2}{n\pi}\left[\cos(n\pi) - \cos 0\right] \\ &= \frac{2}{n\pi}\left[(-1)^n - 1\right] \end{aligned}$$

   The Fourier Sine Series is

$$\boxed{\sum_{n=1}^{\infty} \frac{2}{n\pi}\left[(-1)^n - 1\right]\sin(n\pi x)}$$

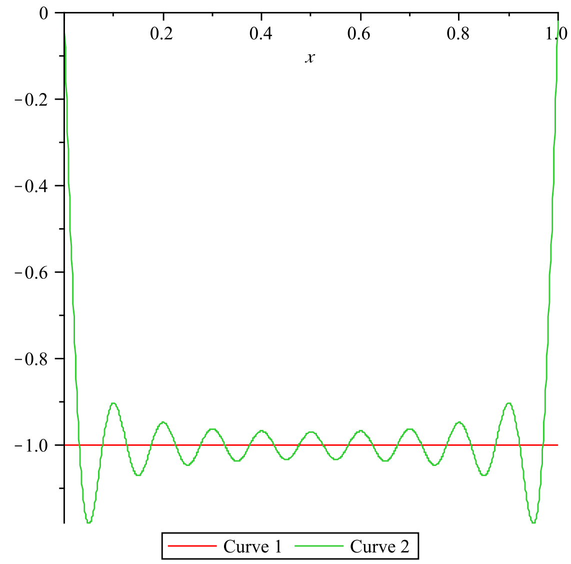



Figure 1: First 20 terms in the Sine Series for Problem 5.

7. For the function $f(x) = x^2$ on $0 < x < \pi$ we have:

$$\begin{aligned}
b_n &= \frac{2}{T}\int_0^T f(x)\sin\left(\frac{n\pi x}{T}\right)\,dx \\
&= \frac{2}{\pi}\int_0^\pi x^2\sin\left(\frac{n\pi x}{\pi}\right)\,dx \\
&= \frac{2}{\pi}\int_0^\pi x^2\sin(nx)\,dx \\
&= \frac{2}{\pi}\left[\frac{2x\sin(nx)}{n^2} - \frac{(n^2x^2-2)\cos(nx)}{n^3}\right]_0^\pi \\
&= \frac{2}{\pi}\left[\left(\frac{2\pi\sin(n\pi)}{n^2} - \frac{(n^2\pi^2-2)\cos(n\pi)}{n^3}\right) - \left(0 - \frac{(0-2)\cos 0}{n^3}\right)\right] \\
&= \frac{2}{\pi}\left[\left(0 - \frac{(n^2\pi^2-2)(-1)^n}{n^3}\right) - \left(0 + \frac{2}{n^3}\right)\right] \\
&= \frac{2}{\pi}\left[\frac{\pi^2(-1)^{n+1}}{n} + \frac{2(-1)^n}{n^3} - \frac{2}{n^3}\right] \\
&= \frac{2\pi}{n}(-1)^{n+1} + \frac{4}{\pi n^3}\left[(-1)^n - 1\right]
\end{aligned}$$

The Fourier Sine Series is

$$\boxed{\sum_{n=1}^{\infty}\left\{\frac{2\pi}{n}(-1)^{n+1} + \frac{4}{\pi n^3}\left[(-1)^n - 1\right]\right\}\sin(nx)}$$

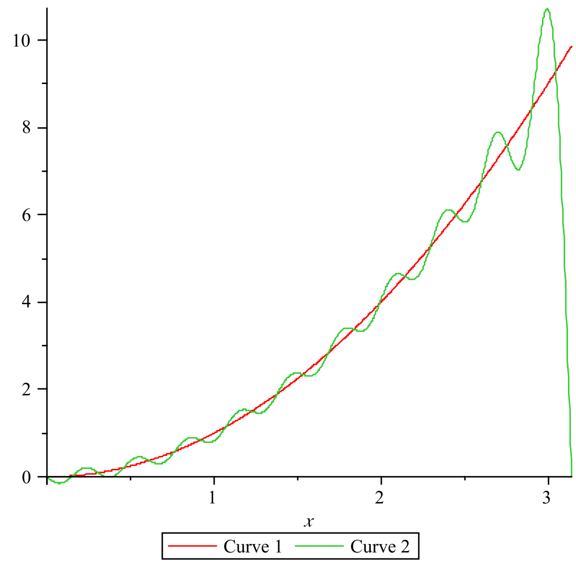



Figure 2: First 20 terms in the Sine Series for Problem 7.

11. For the function $f(x) = \pi - x$ on $0 < x < \pi$, we have:

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^\pi (\pi - x)\,dx \\ &= \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^\pi \\ &= \frac{2}{\pi}\left[\pi^2 - \frac{\pi^2}{2}\right] \\ &= \pi \end{aligned}$$

and

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi (\pi - x)\cos(nx)\,dx \\ &= -\frac{2(-1+\cos(n\pi))}{n^2\pi} \\ &= \frac{2}{n^2\pi}[1-(-1)^n] \end{aligned}$$

The Fourier Cosine Series is:

$$\boxed{\frac{\pi}{2} + \sum_{n=1}^{\infty}\left\{\frac{2}{n^2\pi}[1-(-1)^n]\right\}\cos(nx)}$$

15. For the function $f(x) = \sin x$ on $0 < x < \pi$, we have:

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^\pi \sin x\,dx \\ &= \frac{2}{\pi}\left[-\cos x\right]_0^\pi \\ &= \frac{4}{\pi} \end{aligned}$$

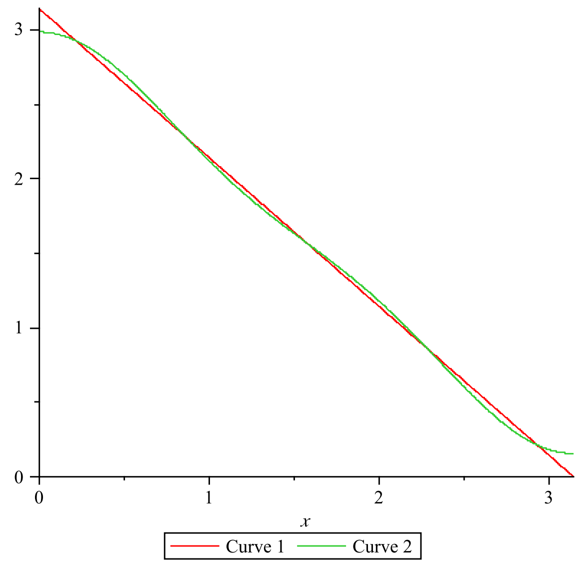



Figure 3: First 3 terms in the Cosine Series for Problem 11.

and

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \, dx = 0$$

and

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} \sin x \cos(nx)\, dx \\ &= \frac{2}{\pi}\left[\frac{1+\cos(n\pi)}{1-n^2}\right] \\ &= \frac{2}{\pi}\left[\frac{1+(-1)^n}{1-n^2}\right], \quad n = 2, 3, 4, \ldots \end{aligned}$$

The Fourier Cosine Series is

$$\boxed{\frac{2}{\pi} + \sum_{n=2}^{\infty}\left\{\frac{2}{\pi}\left[\frac{1+(-1)^n}{1-n^2}\right]\right\}\cos(nx)}$$

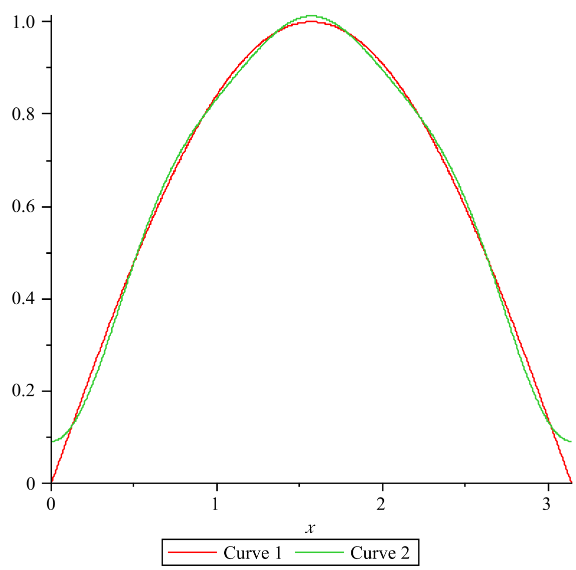



Figure 4: First 6 terms in the Cosine Series for Problem 15.