Math 220 – Section 10.6 Solutions

1. Given that $\alpha = 1$ and $L = 1$ the solution is of the form:

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos n\pi t + b_n \sin n\pi t) \sin n\pi x$$

where the a_n and b_n are found from the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ as follows:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin n\pi x$$
$$g(x) = \sum_{n=1}^{\infty} n\pi b_n \sin n\pi x$$

The a_n are found by finding the Sine Series for $f(x)$:

$$a_n = \frac{2}{1} \int_0^1 x(1-x) \sin n\pi x \, dx$$
$$a_n = \frac{4[1-(-1)^n]}{n^3\pi^3}$$

The b_n are found by finding the Sine Series for $f(x)$:

$$n\pi b_n = \frac{2}{1} \int_0^1 \sin 7\pi x \sin n\pi x \, dx$$
$$n\pi b_n = \begin{cases} 0 & n \neq 7 \\ 1 & n = 7 \end{cases}$$
$$\Rightarrow \quad b_7 = \frac{1}{7\pi} \text{ and } b_n = 0 \text{ for } n \neq 7$$

The solution is then:

$$\boxed{u(x,t) = \frac{1}{7\pi} \sin 7\pi t \sin 7\pi x + \sum_{n=1}^{\infty} \left(\frac{4[1-(-1)^n]}{n^3\pi^3} \right) \cos n\pi t \sin n\pi x}$$

4. Given that $\alpha = 3$ and $L = \pi$ the solution is of the form:

$$u(x,t) = \sum_{n=1}^{\infty} (a_n \cos 3nt + b_n \sin 3nt) \sin nx$$

where the a_n and b_n are found from the initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ as follows:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin nx$$
$$g(x) = \sum_{n=1}^{\infty} 3nb_n \sin nx$$

Using the initial condition $u(x,0) = \sin 4x + 7 \sin 5x$ we find that:

$$a_4 = 4, \; a_5 = 7$$
$$a_n = 0 \text{ for } n \neq 4, 5$$

The other initial condition is:

$$u_t(x,0) = \begin{cases} x, & 0 < x < \pi/x \\ \pi - x, & \pi/x < x < \pi \end{cases}$$

To get the b_n we must find the Sine Series for the function above:

$$3nb_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx\right]$$

$$3nb_n = \frac{4}{n^2\pi} \sin \frac{n\pi}{2}$$

$$b_n = \frac{4}{3\pi n^3} \sin \frac{n\pi}{2}$$

The solution is then:

$$\boxed{u(x,t) = 4\cos 12t \sin 4x + 7\cos 15t \sin 5x + \sum_{n=1}^{\infty} \frac{4}{3\pi n^3} \sin \frac{n\pi}{2} \sin 3nt \sin nx}$$