Math 220 – Section 1.2 Solutions

1. (a) Substituting $\phi(x) = x^2$ for y in the given differential equation, we have:

$$\begin{aligned} x\frac{dy}{dx} &= 2y \\ x\frac{d}{dx}(x^2) &= 2(x^2) \\ x(2x) &= 2x^2 \\ 2x^2 &= 2x^2 \end{aligned}$$

Therefore, since the ODE is satisfied for all x, $\phi(x) = x^2$ is an explicit solution on $(-\infty, \infty)$.

(b) Substituting $\phi(x) = e^x - x$ for y in the given differential equation, we have:

$$\begin{aligned} \frac{dy}{dx} + y^2 &= e^{2x} + (1-2x)e^x + x^2 - 1 \\ \frac{d}{dx}(e^x - x) + (e^x - x)^2 &= e^{2x} + (1-2x)e^x + x^2 - 1 \\ e^x - 1 + e^{2x} - 2xe^x + x^2 &= e^{2x} + (1-2x)e^x + x^2 - 1 \\ e^{2x} + (1-2x)e^x + x^2 - 1 &= e^{2x} + (1-2x)e^x + x^2 - 1 \end{aligned}$$

Therefore, since the ODE is satisfied for all x, $\phi(x) = e^x - x$ is an explicit solution on $(-\infty, \infty)$.

(c) Substituting $\phi(x) = x^2 - x^{-1}$ for y in the given differential equation, we have:

$$\begin{aligned} x^2\frac{d^2y}{dx^2} &= 2y \\ x^2\frac{d^2}{dx^2}(x^2 - x^{-1}) &= 2(x^2 - x^{-1}) \\ x^2(2 - 2x^{-3}) &= 2x^2 - 2x^{-1} \\ 2x^2 - 2x^{-1} &= 2x^2 - 2x^{-1} \end{aligned}$$

Therefore, since the ODE is satisfied and $\phi(x)$ is defined on the interval $(0, \infty)$, $\phi(x) = x^2$ is an explicit solution on $(0, \infty)$.

3. Substituting $y = \sin x + x^2$ into the ODE, we have:

$$\begin{aligned} \frac{d^2y}{dx^2} + y &= x^2 + 2 \\ \frac{d^2}{dx^2}(\sin x + x^2) + (\sin x + x^2) &= x^2 + 2 \\ -\sin x + 2 + \sin x + x^2 &= x^2 + 2 \\ x^2 + 2 &= x^2 + 2 \end{aligned}$$

Since the ODE is satisfied, $y = \sin x + x^2$ is a solution.

5. Substituting $\theta = 2e^{3t} - e^{2t}$ into the ODE, we have:

$$\begin{aligned} \frac{d^2\theta}{dt^2} - \theta\frac{d\theta}{dt} + 3\theta &= -2e^{2t} \\ \frac{d^2}{dt^2}(2e^{3t} - e^{2t}) - (2e^{3t} - e^{2t})\frac{d}{dt}(2e^{3t} - e^{2t}) + 3(2e^{3t} - e^{2t}) &= -2e^{2t} \\ 18e^{3t} - 4e^{2t} - (2e^{3t} - e^{2t})(6e^{3t} - 2e^{2t}) + 6e^{3t} - 3e^{2t} &= -2e^{2t} \\ 18e^{3t} - 4e^{2t} - 12e^{6t} + 10e^{5t} - 2e^{4t} + 6e^{3t} - 3e^{2t} &= -2e^{2t} \\ -12e^{6t} + 10e^{5t} - 2e^{4t} + 24e^{3t} - 7e^{2t} &= -2e^{2t} \end{aligned}$$

Since the ODE is **not** satisfied, $\theta = 2e^{3t} - e^{2t}$ is **not** a solution.

9. Implicitly differentiating the equation $x^2 + y^2 = 4$ we have:

$$\begin{aligned} \frac{d}{dx}(x^2+y^2) &= \frac{d}{dx}(4) \\ 2x + 2y\frac{dy}{dx} &= 0 \\ x + y\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{x}{y} \end{aligned}$$

Therefore, the equation $x^2 + y^2 = 4$ is **not** an implicit solution to $\dfrac{dy}{dx} = \dfrac{x}{y}$.

11. Implicitly differentiating the equation $e^{xy} + y = x - 1$ we have:

$$\begin{aligned} \frac{d}{dx}(e^{xy}+y) &= \frac{d}{dx}(x+1) \\ e^{xy}\left(y + x\frac{dy}{dx}\right) + \frac{dy}{dx} &= 1 \\ \frac{dy}{dx}(xe^{xy}+1) + ye^{xy} &= 1 \\ \frac{dy}{dx} &= \frac{1-ye^{xy}}{1+xe^{xy}} \\ \frac{dy}{dx} &= \frac{e^{-xy}-y}{e^{-xy}+x} \end{aligned}$$

Therefore, the equation $e^{xy} + y = x - 1$ is an implicit solution to $\dfrac{dy}{dx} = \dfrac{e^{-xy}-y}{e^{-xy}+x}$.

20. Substituting $\phi(x) = e^{mx}$ into each ODE and solving we have:

(a)

$$\begin{aligned} \frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y &= 0 \\ \frac{d^2}{dx^2}(e^{mx}) + 6\frac{d}{dx}(e^{mx}) + 5e^{mx} &= 0 \\ m^2e^{mx} + 6me^{mx} + 5e^{mx} &= 0 \\ m^2 + 6m + 5 &= 0 \\ (m+1)(m+5) &= 0 \\ m = -1,\ m &= -5 \end{aligned}$$

(b)

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\begin{aligned}
\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} &= 0 \\
\frac{d^3}{dx^3}(e^{mx}) + 3\frac{d^2}{dx^2}(e^{mx}) + 2\frac{d}{dx}(e^{mx}) &= 0 \\
m^3e^{mx} + 3m^2e^{mx} + 2me^{mx} &= 0 \\
m^3 + 3m^2 + 2m &= 0 \\
m(m^2 + 3m + 2) &= 0 \\
m(m+1)(m+2) &= 0 \\
m = 0,\ m &= -1,\ m = -2
\end{aligned}
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