Math 220 – Section 2.2 Solutions

7. Solve the equation:

$$\frac{dy}{dx} = y(2 + \sin x)$$

Separating variables and integrating, we get:

$$\begin{aligned} \frac{dy}{dx} &= y(2 + \sin x) \\ \frac{dy}{y} &= (2 + \sin x)\,dx \\ \int \frac{dy}{y} &= \int (2 + \sin x)\,dx \\ \Aboxed{\ln|y| &= 2x - \cos x + C} \end{aligned}$$

9. Solve the equation:

$$\frac{dy}{dx} = \frac{1 - x^2}{y^2}$$

Separating variables and integrating, we get:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1 - x^2}{y^2} \\ y^2\,dy &= (1 - x^2)\,dx \\ \int y^2\,dy &= \int (1 - x^2)\,dx \\ \Aboxed{\frac{1}{3}y^3 &= x - \frac{1}{3}x^3 + C} \end{aligned}$$

13. Solve the equation:

$$\frac{dx}{dt} + x^2 = x$$

Separating variables and integrating, we get:

$$\begin{aligned} \frac{dx}{dt} + x^2 &= x \\ \frac{dx}{dt} &= x - x^2 \\ \frac{dx}{x - x^2} &= dt \\ \int \frac{dx}{x(1 - x)} &= \int dt \end{aligned}$$

To integrate the left hand side, we perform partial fraction decomposition:

$$\begin{aligned} \frac{1}{x(1 - x)} &= \frac{A}{x} + \frac{B}{1 - x} \\ 1 &= A(1 - x) + Bx \end{aligned}$$

Letting $x = 0$, we get $A = 1$. Letting $x = 1$, we get $B = 1$. Therefore,

$$\int \frac{dx}{x(1-x)} = \int \left(\frac{1}{x} + \frac{1}{1-x} \right) dx = \ln|x| - \ln|1-x|$$

and the solution to the differential equation is:

$$\boxed{\ln|x| - \ln|1-x| = t + C}$$

Doing a little algebra, we can simplify:

$$\begin{aligned}
\ln\left|\frac{x}{1-x}\right| &= t + C \\
\left|\frac{x}{1-x}\right| &= Ke^t \\
\frac{x}{1-x} &= \pm Ke^t \\
x &= (1-x)(\pm Ke^t) \\
x &= \pm Ke^t \mp xKe^t \\
x \pm xKe^t &= \pm Ke^t \\
x(1 \pm Ke^t) &= \pm Ke^t \\
x(t) &= \frac{Ke^t}{1 \pm Ke^t}
\end{aligned}$$

18. Solve the IVP:

$$\frac{dy}{dx} = (1+y^2)\tan x, \quad y(0) = \sqrt{3}$$

Separating variables and integrating, we have:

$$\begin{aligned}
\frac{dy}{dx} &= (1+y^2)\tan x \\
\frac{dy}{1+y^2} &= \tan x \, dx \\
\int \frac{dy}{1+y^2} &= \int \tan x \, dx \\
\tan^{-1} y &= -\ln|\cos x| + C
\end{aligned}$$

We find C using the IC $y(0) = \sqrt{3}$:

$$\begin{aligned}
\tan^{-1}\sqrt{3} &= \ln|\cos 0| + C \\
\frac{\pi}{3} &= \ln|1| + C \\
C &= \frac{\pi}{3}
\end{aligned}$$

Therefore, the solution is:

$$\boxed{\tan^{-1} y = -\ln|\cos x| + \frac{\pi}{3}}$$

or

$$y(x) = \tan\left(-\ln|\cos x| + \frac{\pi}{3}\right)$$

24. Solve the IVP:

$$\frac{dy}{dx} = 8x^3 e^{-2y}, \quad y(1) = 0$$

Separating variables and integrating, we have:

$$\begin{aligned} \frac{dy}{dx} &= 8x^3 e^{-2y} \\ e^{2y}\, dy &= 8x^3\, dx \\ \int e^{2y}\, dy &= \int 8x^3\, dx \\ \frac{1}{2}e^{2y} &= 2x^4 + C \end{aligned}$$

We find C using the IC $y(1) = 0$:

$$\begin{aligned} \frac{1}{2}e^{2(0)} &= 2(1)^4 + C \\ \frac{1}{2} &= 2 + C \\ C &= -\frac{3}{2} \end{aligned}$$

Therefore, the solution is:

$$\boxed{\frac{1}{2}e^{2y} = 2x^4 - \frac{3}{2}}$$

or

$$y(x) = \frac{1}{2}\ln(4x^4 - 3)$$