Math 220 – Section 2.3 Solutions

9. The ODE:

$$\frac{dr}{d\theta} + (\tan\theta)r = \sec\theta$$

is in standard form because the coefficient of $\frac{dr}{d\theta}$ is 1. Therefore, we have $P(\theta) = \tan\theta$ and $Q(\theta) = \sec\theta$. The integrating factor, $\mu(\theta)$, is then:

$$\mu(\theta) = \exp\left(\int P(\theta)\, d\theta\right) = \exp\left(\int \tan\theta\, d\theta\right) = \exp(\ln|\sec\theta + \tan\theta|) = \sec\theta + \tan\theta$$

Therefore, we have:

$$r(\theta) = \frac{1}{\mu(\theta)}\left[\int \mu(\theta)Q(\theta)\, d\theta + C\right]$$

$$r(\theta) = \frac{1}{\sec\theta + \tan\theta}\left[\int (\sec^2\theta + \sec\theta\tan\theta)\, d\theta + C\right]$$

$$r(\theta) = \frac{1}{\sec\theta + \tan\theta}\left[(\tan\theta + \sec\theta) + C\right]$$

$$\boxed{r(\theta) = 1 + \frac{C}{\sec\theta + \tan\theta}}$$

13. In the equation:

$$y\frac{dx}{dy} + 2x = 5y^3$$

x is the dependent variable while y is the independent variable. So, we need to solve for x as a function of y instead of the other way around like we're used to doing.

First, we must put the ODE in standard form by dividing the entire equation by y:

$$y\frac{dx}{dy} + 2x = 5y^3$$

$$\frac{dx}{dy} + \frac{2}{y}x = 5y^2$$

So $P(y) = \frac{2}{y}$ and $Q(y) = 5y^2$. The integrating factor, $\mu(y)$, is then:

$$\mu(y) = \exp\left(\int \frac{2}{y}\, dy\right) = \exp(2\ln y) = \exp(\ln y^2) = y^2$$

Therefore, the solution is:

$$x(y) = \frac{1}{\mu(y)}\left[\int \mu(y)Q(y)\, dy + C\right]$$

$$x(y) = \frac{1}{y^2}\left[\int (y^2(5y^2)\, dy + C\right]$$

$$x(y) = \frac{1}{y^2}\left[\int 5y^4\, dy + C\right]$$

$$x(y) = \frac{1}{y^2}\left[y^5 + C\right]$$

$$\boxed{x(y) = y^3 + \frac{C}{y^2}}$$

17. To solve the IVP:

$$\frac{dy}{dx} - \frac{y}{x} = xe^x, \quad y(1) = e - 1$$

we identify $P(x) = -\frac{1}{x}$ and $Q(x) = xe^x$. The integrating factor $\mu(x)$ is:

$$\mu(x) = \exp\left(\int P(x)\,dx\right) = \exp\left(-\int \frac{dx}{x}\right) = \exp\left(-\ln x\right) = \exp\left(\ln x^{-1}\right) = x^{-1}$$

The general solution is:

$$\begin{aligned}
y(x) &= \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)\,dx + C\right] \\
y(x) &= \frac{1}{x^{-1}}\left[\int (x^{-1})xe^x\,dx + C\right] \\
y(x) &= x\left[\int e^x\,dx + C\right] \\
y(x) &= x\left[e^x + C\right] \\
y(x) &= xe^x + Cx
\end{aligned}$$

Using the initial condition $y(1) = e - 1$, we find C:

$$\begin{aligned}
y(1) &= e - 1 \\
(1)e^1 + C(1) &= e - 1 \\
e + C &= e - 1 \\
C &= -1
\end{aligned}$$

The solution is:

$$\boxed{y(x) = xe^x - x}$$

18. To solve the IVP:

$$\frac{dy}{dx} + 4y = e^{-x}, \quad y(0) = \frac{4}{3}$$

we identify $P(x) = 4$ and $Q(x) = e^{-x}$. The integrating factor $\mu(x)$ is:

$$\mu(x) = \exp\left(\int P(x)\,dx\right) = \exp\left(\int 4\,dx\right) = \exp\left(4x\right) = e^{4x}$$

The general solution is:

$$\begin{aligned}
y(x) &= \frac{1}{\mu(x)}\left[\int \mu(x)Q(x)\,dx + C\right] \\
y(x) &= \frac{1}{e^{4x}}\left[\int e^{4x}e^{-x}\,dx + C\right] \\
y(x) &= e^{-4x}\left[\int e^{3x}\,dx + C\right] \\
y(x) &= e^{-4x}\left[\frac{1}{3}e^{3x} + C\right] \\
y(x) &= \frac{1}{3}e^{-x} + Ce^{-4x}
\end{aligned}$$

Using the initial condition $y(0) = \frac{4}{3}$, we find C:

$$\begin{aligned} y(0) &= \frac{4}{3} \\ \frac{1}{3}e^{-0} + Ce^{4(0)} &= \frac{4}{3} \\ \frac{1}{3} + C &= \frac{4}{3} \\ C &= 1 \end{aligned}$$

The solution is:

$$\boxed{y(x) = \frac{1}{3}e^{-x} + e^{-4x}}$$

35. (a) $A(t)$ is found by starting with the ODE we derived in class for mixing problems of this type, plugging in the numbers, and solving:

$$\begin{aligned} \frac{dA}{dt} &= r_i c_i - r_o \frac{A}{V_0 + (r_i - r_o)t} \\ \frac{dA}{dt} &= (5)(0.2) - (5)\frac{A}{500 + (5-5)t} \\ \frac{dA}{dt} &= 1 - \frac{A}{100} \\ \frac{dA}{dt} + \frac{1}{100}A = 1 \end{aligned}$$

This equation is in standard form with $P(t) = \dfrac{1}{100}$ and $Q(t) = 1$. Therefore, the integrating factor $\mu(t)$ is:

$$\mu(t) = \exp\left(\int P(t)\,dt\right) = \exp\left(\int \frac{1}{100}\,dt\right) = e^{t/100}$$

Therefore, the solution is:

$$\begin{aligned} A(t) &= \frac{1}{\mu(t)}\left[\int \mu(t)Q(t)\,dt + C\right] \\ A(t) &= e^{-t/100}\left[\int e^{t/100}\,dt + C\right] \\ A(t) &= e^{-t/100}\left[100e^{t/100} + C\right] \\ A(t) &= 100 + Ce^{-t/100} \end{aligned}$$

Find C using the initial condition $A(0) = 5$:

$$\begin{aligned} 5 &= 100 + Ce^{-0/100} \\ C &= -95 \end{aligned}$$

The solution is then:

$$A(t) = 100 - 95e^{-t/100}$$

After 10 min we have $A(10) = 100 - 95e^{-10/100} = 14.0404$ kg. The concentration is then:

$$\boxed{c = \frac{\text{mass}}{\text{volume}} = \frac{14.0404 \text{ kg}}{500 \text{ L}} = 0.281 \frac{\text{kg}}{\text{L}}}$$

(b) Again, we solve by starting with the generic ODE for mixing problems, plugging in numbers, and solving:

$$\begin{aligned}
\frac{dA}{dt} &= r_i c_i - r_o \frac{A}{V_0 + (r_i - r_o)t} \\
\frac{dA}{dt} &= (5)(0.2) - (6)\frac{A}{500 + (5-6)t} \\
\frac{dA}{dt} &= 1 - \frac{6A}{500 - t} \\
\frac{dA}{dt} + \frac{6}{500-t}A &= 1
\end{aligned}$$

The integrating factor for this equation is:

$$\mu(t) = \exp\left(\int \frac{6}{500-t}\, dt\right) = \exp\left(-6\ln(500-t)\right) = (500-t)^{-6}$$

Therefore, the solution is:

$$\begin{aligned}
A(t) &= \frac{1}{\mu(t)}\left[\int \mu(t)(1)\, dt + C\right] \\
A(t) &= \frac{1}{(500-t)^{-6}}\left[\int (500-t)^{-6}\, dt + C\right] \\
A(t) &= (500-t)^6\left[\frac{1}{5}(500-t)^{-5} + C\right] \\
A(t) &= \frac{1}{5}(500+t) + C(500-t)^6
\end{aligned}$$

Find C using the initial condition $A(0) = 14.0404$ kg:

$$\begin{aligned}
14.0404 &= \frac{1}{5}(500) + C(500)^6 \\
C &= (500)^{-6}(14.0404 - 100)
\end{aligned}$$

After 20 min, we then have:

$$\begin{aligned}
A(20) &= \frac{1}{5}(500-20) + C(500-20)^6 \\
A(20) &= 28.7145 \text{ kg}
\end{aligned}$$

The concentration is then:

$$c = \frac{\text{mass}}{\text{volume}} = \frac{28.7145 \text{ kg}}{500 - 20 \text{ L}} = \boxed{0.0598\ \frac{\text{kg}}{\text{L}}}$$