Math 220 – Section 2.4 Solutions

9. To determine if the equation

$$(2xy + 3)\,dx + (x^2 - 1)\,dy = 0$$

is exact, we must determine whether

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Given that $M(x, y) = 2xy + 3$ and $N(x, y) = x^2 - 1$ we have:

$$\frac{\partial M}{\partial y} = 2x, \;\; \frac{\partial N}{\partial x} = 2x$$

Therefore, the equation is exact.

To solve the equation, we must find the function $F(x, y)$ such that

$$\frac{\partial F}{\partial x} = M(x, y), \;\; \frac{\partial F}{\partial y} = N(x, y)$$

To start, we will integrate the first of the above two equations with respect to x:

$$\begin{aligned} \frac{\partial F}{\partial x} &= M(x, y) \\ \frac{\partial F}{\partial x} &= 2xy + 3 \\ \int \frac{\partial F}{\partial x}\,dx &= \int (2xy + 3)\,dx \\ F(x, y) &= x^2y + 3x + g(y) \end{aligned}$$

Now use the second equation to find $g(y)$ by differentiating $F(x, y)$ with respect to y and setting the result equal to $N(x, y)$:

$$\begin{aligned} \frac{\partial F}{\partial y} &= N(x, y) \\ \frac{\partial}{\partial y}(x^2y + 3x + g(y)) &= x^2 - 1 \\ x^2 + 0 + g'(y) &= x^2 - 1 \\ g'(y) &= -1 \\ g(y) &= \int (-1)\,dy \\ g(y) &= -y \end{aligned}$$

Therefore, the function $F(x, y)$ is:

$$F(x, y) = x^2y + 3x - y$$

and the solution to the differential equation is:

$$\boxed{x^2y + 3x - y = C}$$

13. To determine if the equation

$$(1 + \ln y)\, dt + \frac{t}{y}\, dy = 0$$

is exact, we must determine whether

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}$$

Given that $M(t, y) = 1 + \ln y$ and $N(t, y) = \frac{t}{y}$ we have:

$$\frac{\partial M}{\partial y} = \frac{1}{y}, \ \frac{\partial N}{\partial t} = \frac{1}{y}$$

Therefore, the equation is exact.

To solve the equation, we must find the function $F(t, y)$ such that

$$\frac{\partial F}{\partial t} = M(t, y), \ \frac{\partial F}{\partial y} = N(t, y)$$

To start, we will integrate the first of the above two equations with respect to t:

$$\begin{aligned}
\frac{\partial F}{\partial t} &= M(t, y) \\
\frac{\partial F}{\partial t} &= 1 + \ln y \\
\int \frac{\partial F}{\partial t}\, dt &= \int (1 + \ln y)\, dt \\
F(t, y) &= t + t \ln y + g(y)
\end{aligned}$$

Now use the second equation to find $g(y)$ by differentiating $F(t, y)$ with respect to y and setting the result equal to $N(t, y)$:

$$\begin{aligned}
\frac{\partial F}{\partial y} &= N(t, y) \\
\frac{\partial}{\partial y}(t + t \ln y + g(y)) &= \frac{t}{y} \\
0 + \frac{t}{y} + g'(y) &= \frac{t}{y} \\
g'(y) &= 0 \\
g(y) &= 0
\end{aligned}$$

Therefore, the function $F(t, y)$ is:

$$F(t, y) = t + t \ln y$$

and the solution to the differential equation is:

$$\boxed{t + t \ln y = C}$$

17. To determine if the equation

$$\frac{1}{y}\, dx - \left(3y - \frac{x}{y^2}\right) dy = 0$$

is exact, we must determine whether

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Given that $M(x,y) = \frac{1}{y}$ and $N(x,y) = -3y + \frac{x}{y^2}$ we have:

$$\frac{\partial M}{\partial y} = -\frac{1}{y^2}, \; \frac{\partial N}{\partial x} = \frac{1}{y^2}$$

Therefore, the equation is not exact.

21. To solve the IVP:

$$\left(\frac{1}{x} + 2y^2 x\right) dx + (2yx^2 - \cos y)\, dy = 0, \quad y(1) = \pi$$

we first show that the ODE is exact. Since $M(x,y) = \frac{1}{x} + 2y^2 x$ and $N(x,y) = 2yx^2 - \cos y$, we have:

$$\frac{\partial M}{\partial y} = 4yx = \frac{\partial N}{\partial x}$$

The function $F(x,y)$ that satisfies $F_x = M$ and $F_y = N$ is:

$$F(x,y) = \ln x + x^2 y^2 + \sin y$$

Therefore, the general solution is:

$$\ln x + x^2 y^2 + \sin y = C$$

Now use $y(1) = \pi$ to solve for C:

$$C = \ln 1 + (1)^2(\pi)^2 + \sin \pi = \pi^2$$

Therefore, the solution is:

$$\boxed{\ln x + x^2 y^2 + \sin y = \pi^2}$$

25. To determine if the equation

$$\frac{1}{y}\, dx - \left(3y - \frac{x}{y^2}\right) dy = 0$$

is exact, we must determine whether

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Given that $M(x,y) = \frac{1}{y}$ and $N(x,y) = -3y + \frac{x}{y^2}$ we have:

$$\frac{\partial M}{\partial y} = -\frac{1}{y^2}, \; \frac{\partial N}{\partial x} = \frac{1}{y^2}$$

Therefore, the equation is not exact.

21. To solve the IVP:

$$\left(y^2 \sin x\right) dx + \left(\frac{1}{x} - \frac{y}{x}\right) dy = 0, \quad y(\pi) = 1$$

we first check for exactness. Since $M(x,y) = y^2 \sin x$ and $N(x,y) = \frac{1}{x} - \frac{y}{x}$, we have:

$$\frac{\partial M}{\partial y} = 2y \sin x, \; \frac{\partial N}{\partial x} = -\frac{1}{x^2} + \frac{y}{x^2}$$

Therefore, the equation is not exact.

Let's rewrite the ODE:

$$\begin{aligned}
\left(y^2 \sin x\right)\, dx + \left(\frac{1}{x} - \frac{y}{x}\right) dy &= 0 \\
\left(\frac{1}{x} - \frac{y}{x}\right) dy &= -(y^2 \sin x)\, dx \\
\frac{1-y}{x}\, dy &= -y^2 \sin x\, dx \\
\frac{1-y}{y^2}\, dy &= -x \sin x\, dx \\
\int \frac{1-y}{y^2}\, dy &= -\int x \sin x\, dx \\
\int \left(\frac{1}{y^2} - \frac{1}{y}\right) dy &= -\int x \sin x\, dx \\
-\frac{1}{y} - \ln|y| &= x \cos x - \sin x + C
\end{aligned}$$

Use $y(\pi) = 1$ to find C:

$$\begin{aligned}
-\frac{1}{1} - \ln|1| &= \pi \cos \pi - \sin \pi + C \\
-1 - 0 &= -\pi - 0 + C \\
C &= \pi - 1
\end{aligned}$$

Therefore, the solution is:

$$\boxed{-\frac{1}{y} - \ln y = x \cos x - \sin x + \pi - 1}$$