Math 220 – Section 3.2 Solutions

1. The general equation we use for mixing problems is:

$$\frac{dx}{dt} = r_i c_i - r_o \frac{x}{V_0 + (r_i - r_o)t}$$

In this problem,

$$r_i = r_o = 8\ \frac{\text{L}}{\text{min}},\ c_i = 0.05\ \frac{\text{kg}}{\text{L}},\ V_0 = 100\ \text{L}$$

After simplifying, the differential equation is:

$$\frac{dx}{dt} + 0.08x = 0.4$$

This equation is first order, linear, and separable. Therefore, we can use two methods to solve it. Let's compute the integrating factor:

$$\mu(t) = \exp\left(\int 0.08\, dt\right) = \exp(0.08t) = e^{0.08t}$$

The solution is:

$$x(t) = \frac{1}{\mu(t)}\left[\int \mu(t)Q(t) + C\right]$$
$$x(t) = e^{-0.08t}\left[\int 0.4e^{0.08t}\, dt + C\right]$$
$$x(t) = 5 + Ce^{-0.08t}$$

The initial condition is $x(0) = 0.5$ kg. Using this to solve for C we get:

$$0.5 = 5 + Ce^{-0.08(0)}$$
$$C = -4.5$$

Therefore, the mass of salt as a function of time is:

$$x(t) = 5 - 4.5e^{-0.08t}$$

The concentration of salt is:

$$c = \frac{\text{mass}}{\text{volume}}$$

To determine when $c = 0.02$ kg/L, we must solve the following equation for t:

$$0.02 = \frac{x(t)}{V(t)}$$
$$0.02 = \frac{5 - 4.5e^{-0.08t}}{100}$$

Using some algebra, we find that:

$$t = -\frac{\ln \frac{3}{4.5}}{0.08} = 5.07\ \text{min}$$

3. This was done in class. Check your notes for the solution.

4. The general equation we use for mixing problems is:

$$\frac{dx}{dt} = r_i c_i - r_o \frac{x}{V_0 + (r_i - r_o)t}$$

In this problem,

$$r_i = 4\ \frac{\text{L}}{\text{min}},\ r_o = 3\ \frac{\text{L}}{\text{min}},\ c_i = 0.2\ \frac{\text{kg}}{\text{L}},\ V_0 = 100\ \text{L}$$

After simplifying, the differential equation is:

$$\frac{dx}{dt} + \frac{3}{100+t}x = 0.8$$

This equation is first order and linear. Let's compute the integrating factor:

$$\mu(t) = \exp\left(\int \frac{3}{100+t}\,dt\right) = \exp\left(3\ln(100+t)\right) = (100+t)^3$$

The solution is:

$$\begin{aligned}
x(t) &= \frac{1}{\mu(t)}\left[\int \mu(t)Q(t) + C\right] \\
x(t) &= (100+t)^{-3}\left[\int 0.8(100+t)^3\,dt + C\right] \\
x(t) &= (100+t)^{-3}\left[0.2(100+t)^4 + C\right] \\
x(t) &= 0.2(100+t) + C(100+t)^{-3}
\end{aligned}$$

The initial condition is $x(0) = 0$ kg since the tank initially contained pure water. Using this to solve for C we get:

$$\begin{aligned}
0 &= 0.2(100+0) + C(100+0)^{-3} \\
0 &= 20 + \frac{C}{100^3} \\
C &= -20(100)^3
\end{aligned}$$

Therefore, the mass of salt as a function of time is:

$$x(t) = 0.2(100+t) - 20\left(\frac{100}{100+t}\right)^3$$

The concentration of salt is:

$$c = \frac{\text{mass}}{\text{volume}}$$

To determine when $c = 0.1$ kg/L, we must solve the following equation for t:

$$\begin{aligned}
0.1 &= \frac{x(t)}{V(t)} \\
0.1V(t) &= x(t) \\
0.1(100+t) &= 0.2(100+t) - 20\left(\frac{100}{100+t}\right)^3
\end{aligned}$$

After some algebra, we find that:

$$t = 100(\sqrt[4]{2} - 1) = 18.92\ \text{min}$$