Math 220 – Section 4.1 Solutions

3. To verify that $y = \sin 3t + 2\cos 3t$ is a solution to the IVP

$$2y'' + 18y = 0,\ \ y(0) = 2,\ \ y'(0) = 3$$

we plug it into the ODE and check the initial conditions.

First, the two derivatives of y are:

$$\begin{aligned} y' &= 3\cos 3t - 6\sin 3t \\ y'' &= -9\sin 3t - 18\cos 3t \end{aligned}$$

Plugging these into the ODE, we get:

$$\begin{aligned} 2y'' + 18y &= 0 \\ 2(-9\sin 3t - 18\cos 3t) + 18(\sin 3t + 2\cos 3t) &= 0 \\ -18\sin 3t - 36\cos 3t + 18\sin 3t + 36\cos 3t &= 0 \\ 0 &= 0 \end{aligned}$$

Plugging in $t = 0$, we get:

$$\begin{aligned} y(0) &= \sin 3(0) + 2\cos 3(0) = 0 + 2(1) = 2 \\ y'(0) &= 3\cos 3(0) - 6\sin 3(0) = 3(1) - 6(0) = 3 \end{aligned}$$

The amplitude of $y(t) = A\cos\omega t + B\sin\omega t$ is $\sqrt{A^2 + B^2}$. Therefore, the maximum of $|y(t)|$ is:

$$\boxed{\text{maximum } |y(t)| = \sqrt{A^2 + B^2} = \sqrt{2^2 + 1^2} = \sqrt{5}}$$

5. To verify that $y(t) = e^{-3t}\sin(\sqrt{3}t)$ is a solution to:

$$y'' + 6y' + 12y = 0$$

we compute the first two derivatives:

$$\begin{aligned} y' &= \sqrt{3}e^{-3t}\cos(\sqrt{3}t) - 3e^{-3t}\sin(\sqrt{3}t) \\ &= e^{-3t}[\sqrt{3}\cos(\sqrt{3}t) - 3\sin(\sqrt{3}t)] \\ y'' &= e^{-3t}[-3\sin(\sqrt{3}t) - 3\sqrt{3}\cos(\sqrt{3})] - 3e^{-3t}[\sqrt{3}\cos(\sqrt{3}t) - 3\sin(\sqrt{3}t)] \\ &= e^{-3t}[6\sin(\sqrt{3}t) - 6\sqrt{3}\cos(\sqrt{3}t)] \end{aligned}$$

and plug them into the ODE:

$$\begin{aligned} y'' + 6y' + 12y &= 0 \\ e^{-3t}[6\sin(\sqrt{3}t) - 6\sqrt{3}\cos(\sqrt{3}t)] + 6e^{-3t}[\sqrt{3}\cos(\sqrt{3}t) - 3\sin(\sqrt{3}t)] + 12e^{-3t}\sin(\sqrt{3}t) &= 0 \\ 6\sin(\sqrt{3}t) - 6\sqrt{3}\cos(\sqrt{3}t) + 6\sqrt{3}\cos(\sqrt{3}t) - 18\sin(\sqrt{3}t) + 12\sin(\sqrt{3}t) &= 0 \\ 0 &= 0 \end{aligned}$$

As $t \to \infty$,

$$\boxed{\lim_{t\to\infty} y(t) = 0}$$

7. Plugging $y = A\cos 5t + B\sin 5t$ into the ODE:

$$y'' + 2y' + 5y = -50\sin 5t$$

we get:

$$\begin{aligned}(-25A\cos 5t - 25B\sin 5t) + 2(-5A\sin t + 5B\cos 5t) + 5(A\cos 5t + B\sin 5t) &= -50\sin 5t\\ (-25A + 10B + 5A)\cos 5t + (-25B - 10A + 5B)\sin 5t &= -50\sin 5t\end{aligned}$$

Equating the coefficients of $\cos 5t$ and $\sin 5t$ on each side of the equation, we get the following system of equations:

$$\begin{aligned}-25A + 10B + 5A &= 0\\ -25B - 10A + 5B &= -50\end{aligned}$$

The solution is $A = 1$, $B = 2$. Therefore, a synchronous solution is:

$$\boxed{y = \cos 5t + 2\sin 5t}$$