Math 220 – Section 4.4 Solutions

11. The auxiliary equation for $2z'' + z = 9e^{2t}$ is $2r^2 + 1 = 0$. Its roots are $r = \pm\frac{i}{\sqrt{2}}$. Therefore, the homogeneous solution is:

$$z_h(t) = c_1 \cos\left(\frac{t}{\sqrt{2}}\right) + c_2 \sin\left(\frac{t}{\sqrt{2}}\right)$$

The form of the particular solution and its first two derivatives are:

$$\begin{aligned} z_p(t) &= Ae^{2t} \\ z_p'(t) &= 2Ae^{2t} \\ z_p''(t) &= 4Ae^{2t} \end{aligned}$$

Plugging these into the ODE we get:

$$\begin{aligned} 2z'' + z &= 9e^{2t} \\ 2(4Ae^{2t}) + Ae^{2t} &= 9e^{2t} \\ 9Ae^{2t} &= 9e^{2t} \\ 9A &= 9 \\ A &= 1 \end{aligned}$$

Therefore, the particular solution is:

$$\boxed{z_p(t) = e^{2t}}$$

12. The auxiliary equation for $2x' + x = 3t^2$ is $2r + 1 = 0$. Its root is $r = -\frac{1}{2}$. Therefore, the homogeneous solution is:

$$x_h(t) = ce^{-t/2}$$

The form of the particular solution and its first derivative are:

$$\begin{aligned} x_p(t) &= A + Bt + Ct^2 \\ x_p'(t) &= B + 2Ct \end{aligned}$$

Plugging these into the ODE we get:

$$\begin{aligned} 2x' + x &= 3t^2 \\ 2(B + 2Ct) + (A + Bt + Ct^2) &= 3t^2 \\ (2B + A) + t(4C + B) + t^2(C) &= 3t^2 \end{aligned}$$

Equating the coefficients of t^n on both sides, we get the following system of equations:

$$\begin{array}{llcl} t^2: & C & = & 3 \\ t^1: & 4C + B & = & 0 \\ t^0: & 2B + A & = & 0 \end{array}$$

The solution is $C = 3$, $B = -12$, and $A = 24$. Therefore, the particular solution is:

$$\boxed{x_p(t) = 24 - 12t + 3t^2}$$

24. The auxiliary equation for $y'' + y = 4x\cos x$ is $r^2 + 1 = 0$. Its roots are $r = \pm i$. Therefore, the homogeneous solution is:

$$y_h(x) = c_1 \cos x + c_2 \sin x$$

The form of the particular solution and its first two derivatives are:

$$\begin{aligned}
y_p(x) &= x^1[(A_0 + A_1x)\sin x + (B_0 + B_1x)\cos x] \\
&= (A_0x + A_1x^2)\sin x + (B_0x + B_1x^2)\cos x \\
y_p'(x) &= (A_0x + A_1x^2)\cos x + (A_0 + 2A_1x)\sin x - (B_0x + B_1x^2)\sin x + (B_0 + 2B_1x)\cos x \\
&= (B_0 + (A_0 + 2B_1)x + A_1x^2)\cos x + (A_0 + (2A_1 - B_0)x - B_1x^2)\sin x \\
y_p''(x) &= -(B_0 + (A_0 + 2B_1)x + A_1x^2)\sin x + (A_0 + 2B_1 + 2A_1x)\cos x + \\
&\quad\; (A_0 + (2A_1 - B_0)x - B_1x^2)\cos x + (2A_1 - B_0 - 2B_1x)\sin x \\
&= (2A_1 - 2B_0 - (A_0 + 4B_1)x - A_1x^2)\sin x + (2A_0 + 2B_1 + (4A_1 - B_0)x - B_1x^2)\cos x
\end{aligned}$$

Note that we need an extra x^1 in the particular solution because $0+i$ is a root of the auxiliary equation. Plugging these into the ODE we get:

$$\begin{aligned}
y'' + y &= 4x\cos x \\
(2A_1 - 2B_0 - (A_0 + 4B_1)x - A_1x^2)\sin x + (2A_0 + 2B_1 + (4A_1 - B_0)x - B_1x^2)\cos x & \\
+(A_0x + A_1x^2)\sin x + (B_0x + B_1x^2)\cos x &= 4x\cos x \\
(2A_1 - 2B_0)\sin x + (-4B_1)x\sin x + (2A_0 + 2B_1)\cos x + (4A_1)x\cos x &= 4x\cos x
\end{aligned}$$

Equating the coefficients of $x^n \sin x$ and $x^n \cos x$ on both sides, we get the following system of equations:

$$\begin{array}{rrcl}
x\sin x\text{:} & -4B_1 & = & 0 \\
\sin x\text{:} & 2A_1 - 2B_0 & = & 0 \\
x\cos x\text{:} & 4A_1 & = & 4 \\
\cos x\text{:} & 2A_0 + 2B_1 & = & 0
\end{array}$$

The solution is $A_1 = 1$, $B_1 = 0$, $A_0 = 0$, and $B_0 = 1$. Therefore, the particular solution is:

$$\boxed{y_p(x) = x^2 \sin x + x\cos x}$$

27. The homogeneous solution for $y'' + 9y = 4t^3 \sin 3t$ is:

$$y_h(t) = c_1 \cos 3t + c_2 \sin 3t$$

The form of the particular solution is then:

$$\boxed{y_p(t) = t^1[(A_0 + A_1t + A_2t^2 + A_3t^3)\sin 3t + (B_0 + B_1t + B_2t^2 + B_3t^3)\cos 3t]}$$

Note that we need an extra t^1 in the particular solution because $0 + 3i$ is a root of the auxiliary equation.

28. The form of the particular solution to $y'' + 3y' - 7y = t^4e^t$ is:

$$\boxed{y_p(t) = (A + Bt + Ct^2 + Dt^3 + Et^4)e^t}$$

30. The form of the particular solution to $y'' - 2y' + y = 7e^t \cos t$ is:

$$\boxed{y_p(t) = e^t(A\sin t + B\cos t)}$$