Math 220 – Section 4.5 Solutions

5. Given $\theta'' - \theta' - 2\theta = 1 - 2t$ and its particular solution $\theta_p(t) = t - 1$, the general solution is:

$$\theta(t) = \theta_h(t) + \theta_p(t)$$

$$\boxed{\theta(t) = c_1 e^{2t} + c_2 e^{-t} + t - 1}$$

7. Given $y'' = 2y' - y + 2e^x$ and its particular solution $y_p(x) = x^2 e^x$, the general solution is:

$$y(x) = y_h(x) + y_p(x)$$

$$\boxed{y(x) = c_1 e^x + c_2 x e^x + x^2 e^x}$$

20. The homogeneous solution to $y'' + 4y = \sin\theta - \cos\theta$ is

$$y_h(\theta) = c_1 \sin 2\theta + c_2 \cos 2\theta$$

The form of the particular solution and its derivatives are

$$\begin{aligned} y_p(\theta) &= A\sin\theta + B\cos\theta \\ y_p'(\theta) &= A\cos\theta - B\sin\theta \\ y_p''(\theta) &= -A\sin\theta - B\cos\theta \end{aligned}$$

Plugging these into the ODE we have

$$\begin{aligned} y'' + 4y &= \sin\theta - \cos\theta \\ -A\sin\theta - B\cos\theta + 4(A\sin\theta + B\cos\theta) &= \sin\theta - \cos\theta \\ 3A\sin\theta + 3B\cos\theta &= \sin\theta - \cos\theta \end{aligned}$$

Equating the coefficients of $\sin\theta$ and $\cos\theta$ on both sides of the equation we get the following system:

$$\begin{aligned} \sin\theta: \quad 3A &= 1 \\ \cos\theta: \quad 3B &= -1 \end{aligned}$$

The solution is $A = \frac{1}{3}$ and $B = -\frac{1}{3}$. Therefore, the general solution is:

$$\boxed{y(\theta) = c_1 \sin 2\theta + c_2 \cos 2\theta + \frac{1}{3}\sin\theta - \frac{1}{3}\cos\theta}$$

28. The initial value problem is:

$$y'' + y' - 12y = e^t + e^{2t} - 1, \quad y(0) = 1, \quad y'(0) = 3$$

The homogeneous solution to the ODE is:

$$y_h(t) = c_1 e^{-4t} + c_2 e^{3t}$$

The form of the particular solution and its derivatives are:

$$\begin{aligned} y_p(t) &= Ae^t + Be^{2t} + C \\ y_p'(t) &= Ae^t + 2Be^{2t} \\ y_p''(t) &= Ae^t + 4Be^{2t} \end{aligned}$$

Plugging these into the ODE we get:

$$\begin{aligned} y'' + y' - 12y &= e^t + e^{2t} - 1 \\ (Ae^t + 4Be^{2t}) + (Ae^t + 2Be^{2t}) - 12(Ae^t + Be^{2t} + C) &= e^t + e^{2t} - 1 \\ -10Ae^t - 6Be^{2t} - 12C &= e^t + e^{2t} - 1 \end{aligned}$$

We must have $A = -\frac{1}{10}$, $B = -\frac{1}{6}$, and $C = \frac{1}{12}$. Therefore, the general solution and its first derivative are:

$$\begin{aligned} y(t) &= c_1 e^{-4t} + c_2 e^{3t} - \frac{1}{10}e^t - \frac{1}{6}e^{2t} + \frac{1}{12} \\ y'(t) &= -4c_1 e^{-4t} + 3c_2 e^{3t} - \frac{1}{10}e^t - \frac{1}{3}e^{2t} \end{aligned}$$

From the initial conditions, we get the following system:

$$\begin{aligned} y(0) &= c_1 + c_2 - \frac{1}{10} - \frac{1}{6} + \frac{1}{12} = 1 \\ y'(0) &= -4c_1 + 3c_2 - \frac{1}{10} - \frac{1}{3} = 3 \end{aligned}$$

The solution is $c_1 = \frac{1}{60}$ and $c_2 = \frac{7}{6}$. Therefore, the solution is:

$$\boxed{y(t) = \frac{1}{60}e^{-4t} + \frac{7}{6}e^{2t} - \frac{1}{10}e^t - \frac{1}{6}e^{2t} + \frac{1}{12}}$$

33. The form of the particular solution to $x'' - x' - 2x = e^t \cos t - t^2 + t + 1$ is:

$$\boxed{x_p(t) = e^t(A\cos t + B\sin t) + C_0 + C_1 t + C_2 t^2}$$

35. The form of the particular solution to $y'' - 4y' + 5y = e^{5t} + t\sin 3t - \cos 3t$ is:

$$\boxed{y_p(t) = Ae^{5t} + (B_0 + B_1 t)\sin 3t + (C_0 + C_1 t)\cos 3t}$$