Math 220 – Section 7.5 Solutions

1. Solve the IVP
$$y'' - 2y' + 5y = 0,\ y(0) = 2,\ y'(0) = 4$$

   Taking the Laplace Transform of the equation we have
$$\begin{aligned} \mathscr{L}\{y'' - 2y' + 5y\} &= \mathscr{L}\{0\} \\ \mathscr{L}\{y''\} - 2\mathscr{L}\{y'\} + 5\mathscr{L}\{y\} &= 0 \\ s^2Y(s) - sy(0) - y'(0) - 2(sY(s) - y(0)) + 5Y(s) &= 0 \\ s^2Y(s) - 2s - 4 - 2(sY(s) - 2) + 5Y(s) &= 0 \\ (s^2 - 2s + 5)Y(s) &= 2s \\ Y(s) &= \frac{2s}{s^2 - 2s + 5} \end{aligned}$$

   Since the denominator has complex roots, we complete the square and manipulate
$$\begin{aligned} Y(s) &= \frac{2s}{s^2 - 2s + 5} \\ &= \frac{2s}{(s-1)^2 + 4} \\ &= \frac{2s - 2 + 2}{(s-1)^2 + 4} \\ &= \frac{2s - 2}{(s-1)^2 + 4} + \frac{2}{(s-1)^2 + 4} \\ &= 2 \cdot \frac{s-1}{(s-1)^2 + 4} + \frac{2}{(s-1)^2 + 4} \end{aligned}$$

   We can now use a table to find the inverse Laplace Transform to get
$$\boxed{y(t) = 2e^t \cos 2t + e^t \sin 2t}$$

3. Solve the IVP
$$y'' + 6y' + 9y = 0,\ y(0) = -1,\ y'(0) = 6$$

   Taking the Laplace Transform of the equation we have
$$\begin{aligned} \mathscr{L}\{y'' + 6y' + 9y\} &= \mathscr{L}\{0\} \\ \mathscr{L}\{y''\} + 6\mathscr{L}\{y'\} + 9\mathscr{L}\{y\} &= 0 \\ s^2Y(s) - sy(0) - y'(0) + 6(sY(s) - y(0)) + 9Y(s) &= 0 \\ s^2Y(s) + s - 6 + 6(sY(s) + 1) + 9Y(s) &= 0 \\ (s^2 + 6s + 9)Y(s) &= -s \\ Y(s) &= -\frac{s}{s^2 + 6s + 9} \\ Y(s) &= -\frac{s}{(s+3)^2} \end{aligned}$$

Using the method of partial fractions we find that

$$Y(s) = -\frac{1}{s+3} + \frac{3}{(s+3)^2}$$

We can now use a table to find the inverse Laplace Transform to get

$$\boxed{y(t) = -e^{-3t} + 3te^{-3t}}$$

5. Solve the IVP

$$w'' + w = t^2 + 2,\ w(0) = 1,\ w'(0) = -1$$

Taking the Laplace Transform of the equation we have

$$\begin{aligned}
\mathscr{L}\{w'' + w\} &= \mathscr{L}\left\{t^2 + 2\right\} \\
\mathscr{L}\{w''\} + \mathscr{L}\{w\} &= \mathscr{L}\left\{t^2\right\} + 2\mathscr{L}\{1\} \\
s^2W(s) - sw(0) - w'(0) + W(s) &= \frac{2!}{s^3} + \frac{2}{s} \\
s^2W(s) - s + 1 + W(s) &= \frac{2}{s^3} + \frac{2}{s} \\
(s^2+1)W(s) &= s - 1 + \frac{2}{s} + \frac{2}{s^3} \\
W(s) &= \frac{s - 1 + \frac{2}{s} + \frac{2}{s^3}}{s^2+1} \\
W(s) &= \frac{s^4 - s^3 + 2s^2 + 2}{s^3(s^2+1)}
\end{aligned}$$

Using the method of partial fractions we find that

$$W(s) = \frac{2}{s^3} + \frac{s-1}{s^2+1}$$

Rewriting the second fraction above we have

$$W(s) = \frac{2}{s^3} + \frac{s}{s^2+1} - \frac{1}{s^2+1}$$

We can now use a table to find the inverse Laplace Transform to get

$$\boxed{w(t) = t^2 + \cos t - \sin t}$$

7. Solve the IVP

$$y'' - 7y' + 10y = 9\cos t + 7\sin t,\ y(0) = 5,\ y'(0) = -4$$

Taking the Laplace Transform of the equation we have

$$\begin{aligned}
\mathscr{L}\{y''-7y'+10y\} &= \mathscr{L}\{9\cos t+7\sin t\}\\
\mathscr{L}\{y''\}-7\mathscr{L}\{y'\}+10\mathscr{L}\{y\} &= 9\mathscr{L}\{\cos t\}+7\mathscr{L}\{\sin t\}\\
s^2Y(s)-sy(0)-y'(0)-7(sY(s)-y(0))+10Y(s) &= \frac{9s}{s^2+1}+\frac{7}{s^2+1}\\
s^2Y(s)-5s+4-7(sY(s)-5)+10Y(s) &= \frac{9s+7}{s^2+1}\\
(s^2-7s+10)Y(s) &= 5s-39+\frac{9s+7}{s^2+1}\\
Y(s) &= \frac{5s-39}{s^2-7s+10}+\frac{9s+7}{(s^2-7s+10)(s^2+1)}\\
Y(s) &= \frac{(5s-39)(s^2+1)+9s+7}{(s^2-7s+10)(s^2+1)}\\
Y(s) &= \frac{5s^3-39s^2+14s-32}{(s^2-7s+10)(s^2+1)}
\end{aligned}$$

Using the method of partial fractions we find that

$$Y(s)=\frac{8}{s-2}-\frac{4}{s-5}+\frac{s}{s^2+1}$$

We can now use a table to find the inverse Laplace Transform to get

$$\boxed{y(t)=8e^{2t}-4e^{5t}+\cos t}$$

9. Solve the IVP

$$z''+5z'-6z=21e^{t-1},\ z(1)=-1,\ z'(1)=9$$

First, we will make the change of variables

$$\tau=t-1$$

So that the IVP turns into

$$\frac{d^2}{d\tau^2}z+5\frac{dz}{d\tau}-6z=21e^{\tau},\ z(0)=-1,\ \frac{dz}{d\tau}(0)=9$$

We'll let $'$ represent differentiation with respect to τ. Then taking the Laplace Transform of the equation we have

$$\begin{aligned}
\mathscr{L}\{z''+5z'-6z\} &= \mathscr{L}\{21e^{\tau}\}\\
\mathscr{L}\{z''\}+5\mathscr{L}\{z'\}-6\mathscr{L}\{z\} &= 21\mathscr{L}\{e^{\tau}\}\\
s^2Z(s)-sz(0)-z'(0)+5(sZ(s)-z(0))-6Z(s) &= \frac{21}{s-1}\\
s^2Z(s)+s-9+5(sZ(s)+1)-6Z(s) &= \frac{21}{s-1}\\
(s^2+5s-6)Z(s) &= -s+4+\frac{21}{s-1}\\
Z(s) &= \frac{-s+4}{s^2+5s-6}+\frac{21}{(s^2+5s-6)(s-1)}\\
Z(s) &= \frac{(-s+4)(s-1)+21}{(s^2+5s-6)(s-1)}\\
Z(s) &= \frac{-s^2+5s+17}{(s^2+5s-6)(s-1)}
\end{aligned}$$

Using the method of partial fractions we find that

$$Z(s) = -\frac{1}{s+6} + \frac{3}{(s-1)^2}$$

We can now use a table to find the inverse Laplace Transform to get

$$z(\tau) = -e^{-6\tau} + 3\tau e^{\tau}$$

Then substituting $\tau = t - 1$ we have

$$\boxed{z(t) = -e^{-6(t-1)} + 3(t-1)e^{t-1}}$$

11. To solve the initial value problem:

$$y'' - y = t - 2,\ y(2) = 3,\ y'(2) = 0$$

we first make the change of variables $\tau = t - 2$ to transform the IVP into

$$y'' - y = \tau,\ y(0) = 3,\ y'(0) = 0$$

where $'$ means $\frac{d}{d\tau}$. Taking the Laplace Transform and solving for $Y(s)$ we have:

$$\begin{aligned}
\mathscr{L}\{y'' - y\} &= \mathscr{L}\{\tau\} \\
\mathscr{L}\{y''\} - \mathscr{L}\{y\} &= \mathscr{L}\{\tau\} \\
s^2Y(s) - sy(0) - y'(0) - Y(s) &= \frac{1}{s^2} \\
s^2Y(s) - 3s - Y(s) &= \frac{1}{s^2} \\
(s^2 - 1)Y(s) &= 3s + \frac{1}{s^2} \\
Y(s) &= \frac{3s + \frac{1}{s^2}}{s^2 - 1} \\
Y(s) &= \frac{3s^3 + 1}{s^2(s^2 - 1)}
\end{aligned}$$

Using the method of partial fractions, we have

$$Y(s) = \frac{2}{s-1} + \frac{1}{s+1} - \frac{1}{s^2}$$

Then the inverse Laplace Transform is

$$y(\tau) = 2e^{\tau} + e^{-\tau} - \tau$$

Substituting $\tau = t - 2$ we have

$$\boxed{y(t) = 2e^{t-2} + e^{-(t-2)} - t + 2}$$

13. The initial value problem is

$$y'' - y' - 2y = -8\cos t - 2\sin t,\ y(\pi/2) = 1,\ y'(\pi/2) = 0$$

Using the change of variables $\tau = t - \pi/2$, the IVP turns into:

$$y'' - y' - 2y = -8\cos\left(\tau + \frac{\pi}{2}\right) - 2\sin\left(\tau + \frac{\pi}{2}\right)$$
$$y'' - y' - 2y = 8\sin\tau - 2\cos\tau,\ y(0) = 1,\ y'(0) = 0$$

After taking the Laplace Transform and solving for $Y(s)$ we get:

$$Y(s) = \frac{s^3 - s^2 - s + 7}{(s^2+1)(s^2 - s - 2)}$$

Using the method of partial fractions, we have

$$Y(s) = \frac{7s - 11}{5(s^2+1)} + \frac{3}{5(s-2)} - \frac{1}{s+1}$$

Then the inverse Laplace Transform is

$$y(\tau) = \frac{7}{5}\cos\tau - \frac{11}{5}\sin\tau + \frac{3}{5}e^{2\tau} - e^{-\tau}$$

Substituting $\tau = t - \pi/2$ we have

$$\boxed{y(t) = \frac{7}{5}\sin t + \frac{11}{5}\cos t + \frac{3}{5}e^{2(t-\pi/2)} - e^{-(t-\pi/2)}}$$

17-19. answers in back of book

22. $Y(s) = \dfrac{2s^3 - 17s^2 + 28s - 13}{(s-1)^2(s^2 - 6s + 5)}$