# Math 220 – Section 7.8 Solutions

3. $\displaystyle\int_{-\infty}^{\infty} (\sin 3t)\delta\left(t - \frac{\pi}{2}\right)\, dt = \sin\left(3 \cdot \frac{\pi}{2}\right) = \boxed{-1}$

5. $\displaystyle\int_{0}^{\infty} e^{-2t}\delta\left(t - 1\right)\, dt = \int_{-\infty}^{\infty} e^{-2t}\delta\left(t - 1\right)\, dt = e^{-2(1)} = \boxed{e^{-2}}$

13. Solve the initial value problem:

$$w'' + w = \delta(t - \pi), \quad w(0) = 0, \quad w'(0) = 0$$

Taking the Laplace Tranform we have:

$$\begin{aligned} \mathscr{L}\left\{w''\right\} + \mathscr{L}\left\{w\right\} &= \mathscr{L}\left\{\delta(t - \pi)\right\} \\ s^2W(s) + W(s) &= e^{-\pi s} \\ W(s) &= \frac{e^{-\pi s}}{s^2 + 1} \end{aligned}$$

Inverting the transform we get

$$\boxed{w(t) = \sin(t - \pi)u(t - \pi) = -(\sin t)u(t - \pi)}$$

15. Solve the initial value problem:

$$y'' + 2y' - 3y = \delta(t - 1) - \delta(t - 2), \quad y(0) = 2, \quad y'(0) = -2$$

Taking the Laplace Tranform we have:

$$\begin{aligned} \mathscr{L}\left\{y''\right\} + 2\mathscr{L}\left\{y'\right\} - 3\mathscr{L}\left\{y\right\} &= \mathscr{L}\left\{\delta(t - 1)\right\} - \mathscr{L}\left\{\delta(t - 2)\right\} \\ s^2Y(s) - 2s + 2 + 2(sY(s) - 2) - 3Y(s) &= e^{-s} - e^{-2s} \\ (s^2 + 2s - 3)Y(s) &= 2s + 2 + e^{-s} - e^{-2s} \\ Y(s) &= \frac{2s + 2}{s^2 + 2s - 3} + \frac{e^{-s} - e^{-2s}}{s^2 + 2s - 3} \end{aligned}$$

We need to perform partial fractions twice:

$$\begin{aligned} \frac{2s + 2}{s^2 + 2s - 3} &= \frac{1}{s + 3} + \frac{1}{s - 1} \\ \frac{1}{s^2 + 2s - 3} &= \frac{-\frac{1}{4}}{s + 3} + \frac{\frac{1}{4}}{s - 1} \end{aligned}$$

Then we have

$$Y(s) = \frac{1}{s + 3} + \frac{1}{s - 1} + \frac{1}{4}\left(\frac{1}{s - 1} - \frac{1}{s + 3}\right)e^{-s} - \frac{1}{4}\left(\frac{1}{s - 1} - \frac{1}{s + 3}\right)e^{-2s}$$

Inverting the transform we get

$$\boxed{y(t) = e^{-3t} + e^{t} + \frac{1}{4}\left[e^{t-1} - e^{-3(t-1)}\right]u(t - 1) - \frac{1}{4}\left[e^{t-2} - e^{-3(t-2)}\right]u(t - 2)}$$

21. Solve the initial value problem:

$$y'' + y = \delta(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 1$$

Taking the Laplace Transform we have:

$$\begin{aligned}
\mathscr{L}\{y''\} + \mathscr{L}\{y\} &= \mathscr{L}\{\delta(t - 2\pi)\} \\
s^2Y(s) - 1 + Y(s) &= e^{-2\pi s} \\
(s^2 + 1)Y(s) &= 1 + e^{-2\pi s} \\
Y(s) &= \frac{1}{s^2 + 1} + \frac{e^{-2\pi s}}{s^2 + 1}
\end{aligned}$$

Inverting the transform we get

$$\boxed{y(t) = \sin t + \sin(t - 2\pi)u(t - 2\pi) = \sin t + (\sin t)u(t - 2\pi)}$$