Math 220 – Section 8.1 Solutions

1. The ODE is $y' = x^2 + y^2$ and the initial condition is $y(0) = 1$. We find $y'(0)$ as follows:

$$\begin{aligned} y' &= x^2 + y^2 \\ \Rightarrow \quad y'(0) &= 0^2 + [y(0)]^2 \\ &= 0 + 1 \\ &= 1 \end{aligned}$$

To find $y''(0)$, we must first compute y'':

$$\begin{aligned} y'' &= (y')' \\ &= (x^2 + y^2)' \\ &= 2x + 2yy' \end{aligned}$$

Therefore, we have:

$$\begin{aligned} y'' &= 2x + 2yy' \\ \Rightarrow \quad y''(0) &= 2(0) + 2y(0)y'(0) \\ &= 0 + 2(1)(1) \\ &= 2 \end{aligned}$$

Therefore, the Taylor polynomial approximation to the solution is:

$$y(x) \approx P_2(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 = \boxed{1 + x + x^2}$$

3. The ODE is $y' = \sin y + e^x$ and the initial condition is $y(0) = 0$. We find $y'(0)$ as follows:

$$\begin{aligned} y' &= \sin y + e^x \\ \Rightarrow \quad y'(0) &= \sin 0 + e^0 \\ &= 0 + 1 \\ &= 1 \end{aligned}$$

To find $y''(0)$, we must first compute y'':

$$\begin{aligned} y'' &= (y')' \\ &= (\sin y + e^x)' \\ &= (\cos y)y' + e^x \end{aligned}$$

Therefore, we have:

$$\begin{aligned} y'' &= (\cos y)y' + e^x \\ \Rightarrow \quad y''(0) &= (\cos y(0))y'(0) + e^0 \\ &= (1)(1) + 1 \\ &= 2 \end{aligned}$$

Therefore, the Taylor polynomial approximation to the solution is:

$$y(x) \approx P_2(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 = \boxed{1 + x + x^2}$$

5. The ODE is $x'' + tx = 0$ and the initial conditions are $x(0) = 1$ and $x'(0) = 0$. We find $x''(0)$ as follows:

$$\begin{aligned} x'' &= -tx \\ \Rightarrow \quad x''(0) &= -(0)x(0) \\ &= 0 \end{aligned}$$

Let's compute x''', $x^{(4)}$, $x^{(5)}$, and $x^{(6)}$:

$$\begin{aligned} x''' &= (x'')' \\ &= (-tx)' \\ &= -x - tx' \\ x^{(4)} &= (x''')' \\ &= (-x - tx')' \\ &= -x' - x' - tx'' \\ &= -2x' - tx'' \\ x^{(5)} &= (x^{(4)})' \\ &= (-2x' - tx'')' \\ &= -2x'' - x'' - tx''' \\ &= -3x'' - tx''' \\ x^{(6)} &= (x^{(5)})' \\ &= (-3x'' - tx''')' \\ &= -3x''' - x''' - tx^{(4)} \\ &= -4x''' - tx^{(4)} \end{aligned}$$

Evaluating each of the above derivatives using $x(0) = 1$, $x'(0) = 0$, and $x''(0) = 0$ we have:

$$\begin{aligned} x'''(0) &= -x(0) - (0)x'(0) = -1 \\ x^{(4)}(0) &= -2x'(0) - (0)x''(0) = -2(0) - 0 = 0 \\ x^{(5)}(0) &= -3x''(0) - (0)x'''(0) = -3(0) - 0 = 0 \\ x^{(6)}(0) &= -4x'''(0) - (0)x^{(4)}(0) = -4(-1) - 0 = 4 \end{aligned}$$

Therefore, the Taylor polynomial approximation to the solution is:

$$\begin{aligned} x(t) \approx P_6(t) &= x(0) + x'(0)t + \frac{x''(0)}{2!}t^2 + \frac{x'''(0)}{3!}t^3 + \frac{x^{(4)}(0)}{4!}t^4 + \frac{x^{(5)}(0)}{5!}t^5 + \frac{x^{(6)}(0)}{6!}t^6 \\ &= \boxed{1 - \frac{1}{3!}t^3 + \frac{4}{6!}t^6} \end{aligned}$$