Math 220 – Section 9.5 Solutions

1. First, we find the eigenvalues of the matrix:

$$\begin{aligned} |A - \lambda I| &= 0 \\ \begin{vmatrix} -4-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} &= 0 \\ (-4-\lambda)(-1-\lambda) - (2)(2) &= 0 \\ \lambda^2 + 5\lambda + 4 - 4 &= 0 \\ \lambda^2 + 5\lambda &= 0 \\ \lambda(\lambda+5) &= 0 \\ \boxed{\lambda = 0,\ \lambda = -5} \end{aligned}$$

Now we find the corresponding eigenvectors for each eigenvalue. Start with $\lambda = 0$:

$$\begin{aligned} (A - 0I)\vec{\mathbf{x}} &= \vec{\mathbf{0}} \\ \begin{bmatrix} -4-0 & 2 \\ 2 & -1-0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \Rightarrow \quad 2x_1 - x_2 &= 0 \end{aligned}$$

We let $x_2 = 2$ so that $x_1 = 1$. Thus, an eigenvector for $\lambda = 0$ is $\boxed{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$.

Now we find an eigenvector for $\lambda = -5$:

$$\begin{aligned} (A + 5I)\vec{\mathbf{x}} &= \vec{\mathbf{0}} \\ \begin{bmatrix} -4+5 & 2 \\ 2 & -1+5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \Rightarrow \quad x_1 + 2x_2 &= 0 \end{aligned}$$

We let $x_2 = 1$ so that $x_1 = -2$. Thus, an eigenvector for $\lambda = -5$ is $\boxed{\begin{bmatrix} -2 \\ 1 \end{bmatrix}}$.

3. First, we find the eigenvalues of the matrix:

$$\begin{aligned} |A - \lambda I| &= 0 \\ \begin{vmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{vmatrix} &= 0 \\ (1-\lambda)(4-\lambda) - (-1)(2) &= 0 \\ \lambda^2 - 5\lambda + 4 + 2 &= 0 \\ \lambda^2 - 5\lambda + 6 &= 0 \\ (\lambda-2)(\lambda-3) &= 0 \\ \boxed{\lambda = 2,\ \lambda = 3} \end{aligned}$$

Now we find the corresponding eigenvectors for each eigenvalue. Start with $\lambda = 2$:

$$(A - 2I)\vec{\mathbf{x}} = \vec{\mathbf{0}}$$
$$\begin{bmatrix} 1-2 & -1 \\ 2 & 4-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \quad x_1 + x_2 = 0$$

We let $x_2 = 1$ so that $x_1 = -1$. Thus, an eigenvector for $\lambda = 2$ is $\boxed{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$.

Now we find an eigenvector for $\lambda = 3$:

$$(A - 3I)\vec{\mathbf{x}} = \vec{\mathbf{0}}$$
$$\begin{bmatrix} 1-3 & -1 \\ 2 & 4-3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \quad 2x_1 + x_2 = 0$$

We let $x_2 = 2$ so that $x_1 = -1$. Thus, an eigenvector for $\lambda = 3$ is $\boxed{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}$.

12. The eigenvalues and eigenvectors of the given matrix are:

$$\lambda = 7, \begin{bmatrix} 1 \\ 2 \end{bmatrix}; \; \lambda = -5, \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

The general solution to $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ is then:

$$\boxed{\vec{\mathbf{x}}(t) = c_1 e^{7t}\begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-5t}\begin{bmatrix} -1 \\ 2 \end{bmatrix}}$$

31. The eigenvalues and eigenvectors of the given matrix are:

$$\lambda = 4, \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \; \lambda = -2, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The general solution to $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ is then:

$$\boxed{\vec{\mathbf{x}}(t) = c_1 e^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

Now use the initial conditions to solve for c_1 and c_2:

$$\vec{\mathbf{x}}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$\Rightarrow \quad 3 = c_1 - c_2$$
$$1 = c_1 + c_2$$

The solution to the above system of equations is $c_1 = -1$ and $c_1 = 2$. Therefore, the solution is:

$$\boxed{\vec{\mathbf{x}}(t) = -e^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2e^{-2t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

32. The eigenvalues and eigenvectors of the given matrix are:

$$\lambda = 3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \ \lambda = 4, \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The general solution to $\overrightarrow{\mathbf{x}}' = A\overrightarrow{\mathbf{x}}$ is then:

$$\boxed{\overrightarrow{\mathbf{x}}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}}$$

Now use the initial conditions to solve for c_1 and c_2:

$$\begin{aligned} \overrightarrow{\mathbf{x}}(0) = \begin{bmatrix} -10 \\ -6 \end{bmatrix} &= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\ \Rightarrow \quad -10 &= c_1 + 3c_2 \\ -6 &= c_1 + 2c_2 \end{aligned}$$

The solution to the above system of equations is $c_1 = 2$ and $c_2 = -4$. Therefore, the solution is:

$$\boxed{\overrightarrow{\mathbf{x}}(t) = 2e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 4e^{4t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}}$$