# Summer, 2007 – Exam 1 Solutions

1. (15 pts) Solve the following initial value problem (leave your answer in implicit form):

$$\frac{dy}{dx} = \frac{y\cos x}{1+2y^2}, \quad y(0) = 1$$

**Solution**: The equation is separable but not linear. So we must solve by separation of variables:

$$\begin{aligned} \frac{dy}{dx} &= \frac{y\cos x}{1+2y^2} \\ \frac{1+2y^2}{y}\,dy &= \cos x\,dx \\ \int \left(\frac{1}{y} + 2y\right)\,dy &= \int \cos x\,dx \\ \ln|y| + y^2 &= \sin x + C \end{aligned}$$

Use the initial condition $y(0) = 1$ to solve for $C$:

$$\begin{aligned} \ln|1| + 1^2 &= \sin 0 + C \\ 0 + 1 &= 0 + C \\ C &= 1 \end{aligned}$$

The solution is:

$$\boxed{\ln|y| + y^2 = \sin x + 1}$$

2. (15 pts) Consider the initial value problem:

$$y' = x^2 + y^2, \quad y(0) = 1$$

Use Euler's method with step size $h = \frac{1}{4}$ to find the approximate value of $y(\frac{1}{2})$.

**Solution**: Here we have $x_0 = 0$, $y_0 = 1$, $f(x,y) = x^2 + y^2$, and $h = \frac{1}{4}$. The first step of Euler's Method gives us:

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) \\ &= 1 + \left(\frac{1}{4}\right)(0^2 + 1^2) \\ &= 1 + \frac{1}{4} \\ &= \frac{5}{4} \\ x_1 &= x_0 + h \\ &= 0 + \frac{1}{4} \\ &= \frac{1}{4} \end{aligned}$$

One more step gives us:

$$\begin{aligned}
y_2 &= y_1 + hf(x_1, y_1) \\
&= \frac{5}{4} + \left(\frac{1}{4}\right)\left[\left(\frac{1}{4}\right)^2 + \left(\frac{5}{4}\right)^2\right] \\
&= \frac{5}{4} + \left(\frac{1}{4}\right)\left(\frac{26}{16}\right) \\
&= \frac{106}{64} \\
x_2 &= x_1 + h \\
&= \frac{1}{4} + \frac{1}{4} \\
&= \frac{1}{2}
\end{aligned}$$

The approximate value of $y(\frac{1}{2})$ is $\boxed{\frac{106}{64} = \frac{53}{32} = 1.65625}$.

3. (25 pts) A salt water solution with a concentration of 1 kg/L flows at a constant rate of 2 L/min into a large tank that initially holds 100 L of pure water. The solution inside the tank is kept well-stirred and flows out of the tank at a rate of 4 L/min.

   (a) Determine $V(t)$, the volume of solution in the tank at time $t$.

   (b) At what time will the tank be empty?

   (c) Set up and solve the initial value problem (ODE + initial condition) that governs how much salt $x(t)$ is in the tank at time $t$.

**Solution**:

(a) The volume is given by:

$$\boxed{V(t) = V_0 + (r_i - r_o)t = 100 + (2 - 4)t = 100 - 2t}$$

(b) The tank will be empty when $V(t) = 0$:

$$0 = 100 - 2t \quad \Rightarrow \quad \boxed{t = 50 \text{ min}}$$

(c) The ODE that governs $x(t)$ is:

$$\begin{aligned}
\frac{dx}{dt} &= \text{rate in} - \text{rate out} \\
\frac{dx}{dt} &= r_i c_i - r_o c_o \\
\frac{dx}{dt} &= (2)(1) - (4)\frac{x}{100 - 2t} \\
\frac{dx}{dt} &= 2 - \frac{2}{50 - t}x \\
\frac{dx}{dt} + \frac{2}{50 - t}x &= 2
\end{aligned}$$

The ODE is linear but not separable. Therefore, we solve by first computing the integrating factor:

$$\begin{aligned}\mu(t) &= \exp\left(\int \frac{2}{50-t}\,dt\right)\\ &= \exp\left(-2\ln(50-t)\right)\\ &= \exp\left(\ln(50-t)^{-2}\right)\\ &= (50-t)^{-2}\end{aligned}$$

Multiplying the ODE by $\mu(t)$ and simplifying we get:

$$\begin{aligned}\frac{dx}{dt}(0-t)^{-2} + \frac{2}{50-t}(50-t)^{-2}x &= 2(50-t)^{-2}\\ \frac{d}{dt}[(50-t)^{-2}x(t)] &= 2(50-t)^{-2}\\ \int d[(50-t)^{-2}x(t)] &= \int 2(50-t)^{-2}\,dt\\ (50-t)^{-2}x(t) &= \frac{2}{50-t} + C\end{aligned}$$

We use the initial condition, $x(0) = 0$, to find $C$:

$$\begin{aligned}(50-0)^{-2}(0) &= \frac{2}{50-0} + C\\ 0 &= \frac{1}{25} + C\\ C &= -\frac{1}{25}\end{aligned}$$

Thus, the solution is:

$$\boxed{x(t) = 2(50-t) - \frac{1}{25}(50-t)^2}$$

4. (20 pts) Complete each of the following:

   (a) Find the general solution to: $y'' + 5y' + 6y = 0$.

   (b) Write the form of the particular solution to:

   $$y'' + 5y' + 6y = x^3 + 2x + 3e^x$$

   (**Do not solve for the coefficients!**)

**Solution**:

(a) The auxiliary equation is $r^2 + 5r + 6 = 0$. Its solutions are $x = -3, -2$. Therefore, the general solution is:

$$\boxed{y(x) = c_1e^{-3x} + c_2e^{-2x}}$$

(b) From the method of undetermined coefficients, the form of the particular solution is:

$$\boxed{y_p(x) = A_3x^3 + A_2x^2 + A_1x + A_0 + Be^x}$$

5. (25 pts) Find the general solution to the following nonhomogeneous ODE:

$$y'' + 9y = \cos 3x.$$

**Solution**: The auxiliary equation is $r^2 + 9 = 0$. Its solutions are $x = \pm 3i$. Therefore, the homogeneous solution is:

$$y_h(x) = c_1 \cos 2x + c_2 \sin 3x$$

To obtain the particular solution, we use the method of undetermined coefficients. The guess for the particular solution in this case is:

$$y_p(x) = x(A\cos 3x + B\sin 3x)$$

where the extra $x$ is included because $\cos 3x$ is part of the homogeneous solution. Taking two derivatives of $y_p(x)$ we get:

$$\begin{aligned} y_p'(x) &= A\cos 3x + B\sin 3x + x(-3A\sin 3x + 3B\cos 3x) \\ y_p''(x) &= -3A\sin 3x + 3B\cos 3x + (-3A\sin 3x + 3B\cos 3x) + x(-9A\cos 3x + 9B\sin 3x) \\ &= -6A\sin 3x + 6B\cos 3x + x(-9A\cos 3x + 9B\sin 3x) \end{aligned}$$

Plugging into the ODE we get:

$$\begin{aligned} y_p'' + y_p &= \cos 3x \\ -6A\sin 3x + 6B\cos 3x + x(-9A\cos 3x + 9B\sin 3x) + 9x(A\cos 3x + B\sin 3x) &= \cos 3x \\ -6A\sin 3x + 6B\cos 3x &= \cos 3x \end{aligned}$$

We obtain the following two equations by equating the coefficients of $\sin 3x$ and $\cos 3x$ on the left and right hand sides of the above equation:

$$\begin{aligned} -6A &= 0 \\ 6B &= 1 \end{aligned}$$

The solution is $A = 0$ and $B = \frac{1}{6}$. The total solution is:

$$\boxed{y(x) = c_1 \cos 3x + c_2 \sin 3x + \frac{1}{6} x \sin 3x}$$