Summer, 2007 – Exam 1 Solutions

1. (20 pts) Find the general solution to:

$$\frac{dx}{dt} = x - y$$
$$\frac{dy}{dt} = 2x + 4y$$

Solution: Let's use the elimination method here. We'll take the first equation and solve for y:

$$y = x - x'$$

Then we'll plug this into the second equation:

$$\begin{aligned} y' &= 2x + 4y \\ (x - x')' &= 2x + 4(x - x') \\ x' - x'' &= 2x + 4x - 4x' \\ x'' - 5x' + 6x &= 0 \end{aligned}$$

The auxiliary equation is $r^2 - 5r + 6 = 0$ and the solutions are $r = 2, 3$. Therefore, the general solution for $x(t)$ is:

$$\boxed{x(t) = c_1 e^{2t} + c_2 e^{3t}}$$

We then obtain $y(t)$ from the first equation:

$$\begin{aligned} y(t) &= x(t) - x'(t) \\ y(t) &= c_1 e^{2t} + c_2 e^{3t} - (c_1 e^{2t} + c_2 e^{3t})' \\ y(t) &= c_1 e^{2t} + c_2 e^{3t} - 2c_1 e^{2t} - 3c_2 e^{3t} \end{aligned}$$
$$\boxed{y(t) = -c_1 e^{2t} - 2c_2 e^{3t}}$$

2. (20 pts) Compute the following expressions:

   (a) $\mathscr{L}\left\{te^t + e^{-3t}\sin t\right\}$

   (b) $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2 + 2s + 5}\right\}$

Solution:

(a) We use the property that $\mathscr{L}\{f(t)e^{at}\} = F(s - a)$ for both terms to get:

$$\boxed{\frac{1}{(s-1)^2} + \frac{1}{(s+3)^2 + 1}}$$

(b) We first complete the square to get:

$$\frac{s}{s^2 + 2s + 5} = \frac{s}{(s+1)^2 + 4}$$

Then we rewrite the numerator to get:

$$\frac{s}{(s+1)^2+4} = \frac{s+1-1}{(s+1)^2+4} = \frac{s+1}{(s+1)^2+4} - \frac{1}{2}\frac{2}{(s+1)^2+4}$$

The inverse Laplace transform of the above expression is:

$$\boxed{e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t}$$

3. (20 pts) Consider the following piecewise-defined function:

$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ t, & 1 < t < 2 \\ 0, & t > 2 \end{cases}$$

(a) Write $f(t)$ in terms of step functions.

(b) Compute $\mathscr{L}\{f(t)\}$.

Solution:

(a) Written in terms of step functions, the function is:

$$\boxed{f(t) = 1 + (t-1)u(t-1) - tu(t-2)}$$

(b) The Laplace Transform of $f(t)$ is:

$$\begin{aligned}\mathscr{L}\{f(t)\} &= \mathscr{L}\{1\} + \mathscr{L}\{(t-1)u(t-1)\} - \mathscr{L}\{tu(t-2)\} \\ &= \frac{1}{s} + \frac{e^{-s}}{s^2} - \mathscr{L}\{(t-2)u(t-2)\} - 2\mathscr{L}\{u(t-2)\} \\ &= \boxed{\frac{1}{s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} - \frac{2e^{-2s}}{s}}\end{aligned}$$

4. (20 pts) Find the solution $x(t)$ to the following initial value problem:

$$x'' - 3x' + 2x = \delta(t-1), \; x(0) = 0, \; x'(0) = 1$$

Solution: We solve by taking the Laplace transform of the equation to obtain $X(s)$:

$$\begin{aligned} x'' - 3x' + 2x &= \delta(t-1) \\ \mathscr{L}\{x''\} - 3\mathscr{L}\{x'\} + 2\mathscr{L}\{x\} &= \mathscr{L}\{\delta(t-1)\} \\ s^2X(s) - sx(0) - x'(0) - 3(sX(s) - x(0)) + 2X(s) &= e^{-s} \\ s^2X(s) - 1 - 3sX(s) + 2X(s) &= e^{-s} \\ X(s)(s^2 - 3s + 2) &= 1 + e^{-s} \\ X(s) &= \frac{1}{s^2-3s+2} + \frac{e^{-s}}{s^2-3s+2} \end{aligned}$$

The solution $x(t)$ is the inverse Laplace transform of $X(s)$. In order to perform this calculation we must use partial fraction decomposition:

$$\frac{1}{s^2-3s+2} = \frac{1}{(s-2)(s-1)} = \frac{1}{s-2} - \frac{1}{s-1}$$

Therefore, we have:

$$x(t) = \mathscr{L}^{-1}\{F(s)\}$$
$$x(t) = \mathscr{L}^{-1}\left\{\frac{1}{s-2} - \frac{1}{s-1}\right\} + \mathscr{L}^{-1}\left\{e^{-s}\left(\frac{1}{s-2} - \frac{1}{s-1}\right)\right\}$$
$$\boxed{x(t) = e^{2t} - e^{t} + \left[e^{2(t-1)} - e^{t-1}\right]u(t-1)}$$

5. (20 pts) Find the Taylor Polynomial of degree 3 that approximates the solution to:

$$y' = xy^2,\ y(0) = 1$$

Solution: The Taylor polynomial of degree 3 is of the form:

$$P_3(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3$$

We're given that $y(0) = 1$. We must now find the remaining derivatives evaluated at $x = 0$. First, let's use the ODE to compute y', y'', and y''':

$$y' = xy^2$$
$$y'' = y^2 + 2xyy'$$
$$y''' = 2yy' + 2yy' + 2xy'y' + 2xyy''$$

Evaluating at $x = 0$ we get:

$$y'(0) = (0)y(0)^2 = 0$$
$$y''(0) = y(0)^2 + 2(0)y(0)y'(0) = 1$$
$$y'''(0) = 2y(0)y'(0) + 2y(0)y'(0) + 2(0)y'(0)y'(0) + 2(0)y(0)y''(0) = 0$$

Therefore, the Taylor polynomial of degree 3 that approximates the solution is:

$$\boxed{y(x) \approx P_3(x) = 1 + 0x + \frac{1}{2}x^2 + 0x^3}$$