Summer, 2007 – Final Exam Solutions

1. (20 pts) Find the **explicit** solution to the following initial value problem:

$$\frac{dy}{dx} + \frac{1}{2x}y = \sqrt{x}, \quad y(0) = 1$$

Solution: The equation is linear but not separable. We must first find the integrating factor:

$$\begin{aligned}
\mu(x) &= \exp\left(\int \frac{1}{2x}\,dx\right) \\
\mu(x) &= \exp\left(\frac{1}{2}\ln x\right) \\
\mu(x) &= \sqrt{x}
\end{aligned}$$

Multiplying the ODE by the integrating factor and simplifying we get:

$$\begin{aligned}
\sqrt{x}\frac{dy}{dx} + \frac{\sqrt{x}}{2x}y &= (\sqrt{x})^2 \\
\frac{d}{dx}[\sqrt{x}y] &= x \\
d[\sqrt{x}y] &= x\,dx \\
\int d[\sqrt{x}y] &= \int x\,dx \\
\sqrt{x}y &= \frac{1}{2}x^2 + C
\end{aligned}$$

We now use the initial condition, $y(0) = 1$, to find C:

$$\begin{aligned}
\sqrt{x}y &= \frac{1}{2}x^2 + C \\
\sqrt{0}(1) &= \frac{1}{2}(0)^2 + C \\
C &= 0
\end{aligned}$$

The explicit solution is:

$$\boxed{y(x) = \frac{1}{2}x^{3/2}}$$

2. (20 pts) Complete each part below:

   (a) Find the general solution to the following second order ODE:

$$y'' - 6y' + 5y = 0$$

   (b) Write the form of the particular solution to the following nonhomogeneous, second order ODE:

$$y'' - 6y' + 5y = x + \sin x$$

   Do not solve for the coefficients!

Solution:

(a) The auxiliary equation is $r^2 - 5r + 6 = 0$ and its roots are $r = 1, 5$. Therefore, the general solution is:

$$\boxed{y(x) = c_1 e^x + c_2 e^{5x}}$$

(b) Using the method of undetermined coefficients, the form of the particular solution is:

$$\boxed{y_p(x) = Ax + B + C\cos x + D\sin x}$$

3. (20 pts) Compute the following expressions:

   (a) $\mathscr{L}\left\{\sin 2t + e^{-3t}t^2\right\}$

   (b) $\mathscr{L}^{-1}\left\{\dfrac{s+1}{s^2 - 3s + 2}\right\}$

Solution:

(a) $\mathscr{L}\left\{\sin 2t + e^{-3t}t^2\right\} = \boxed{\dfrac{2}{s^2+4} + \dfrac{2}{(s+3)^3}}$

(b) We first use partial fraction decomposition:

$$\frac{s+1}{s^2 - 3s + 2} = \frac{3}{s-2} + \frac{-2}{s-1}$$

Then we get:

$$\mathscr{L}^{-1}\left\{\frac{s+1}{s^2 - 3s + 2}\right\} = \mathscr{L}^{-1}\left\{\frac{3}{s-2} - \frac{2}{s-1}\right\} = \boxed{3e^{2t} - 2e^t}$$

4. (20 pts) Find all $\lambda > 0$ for which the boundary value problem:

$$y'' + 4\lambda y' + 5\lambda^2 y = 0, \quad y(0) = 0, \quad y\left(\frac{\pi}{2}\right) = 0$$

has nontrivial solutions, $y(x)$, and find these solutions.

Solution: The auxiliary equation is $r^2 + 4\lambda r + 5\lambda^2 = 0$. Its roots are obtained using the quadratic equation:

$$r = \frac{-4\lambda \pm \sqrt{(4\lambda)^2 - 4(1)(5\lambda^2)}}{2(1)}$$
$$r = \frac{-4\lambda \pm \sqrt{16\lambda^2 - 20\lambda^2}}{2}$$
$$r = \frac{-4\lambda \pm \sqrt{-4\lambda^2}}{2}$$
$$r = \frac{-4\lambda \pm 2\lambda i}{2}$$
$$r = -2\lambda \pm \lambda i$$

The general solution is then:

$$y(x) = e^{-2\lambda x}[c_1\cos(\lambda x) + c_2\sin(\lambda x)]$$

Using the first boundary condition, $y(0) = 0$, we find:

$$\begin{aligned} y(0) = e^{-2\lambda \cdot 0}[c_1 \cos(\lambda \cdot 0) + c_2 \sin(\lambda \cdot 0)] &= 0 \\ (1)[c_1(1) + c_2(0)] &= 0 \\ c_1 &= 0 \end{aligned}$$

Using the second boundary condition, $y\left(\frac{\pi}{2}\right) = 0$, we find:

$$\begin{aligned} y\left(\frac{\pi}{2}\right) = e^{-2\lambda \cdot \frac{\pi}{2}} c_2 \sin\left(\lambda \cdot \frac{\pi}{2}\right) &= 0 \\ e^{-\lambda\pi} c_2 \sin\left(\frac{\lambda\pi}{2}\right) &= 0 \\ \sin\left(\frac{\lambda\pi}{2}\right) &= 0 \\ \frac{\lambda\pi}{2} &= n\pi, \quad \text{where } \boxed{n = 1, 2, 3, \ldots} \\ \boxed{\lambda = 2n} \end{aligned}$$

The corresponding solutions are:

$$\boxed{y(x) = e^{-4nx} \sin(2nx)}$$

5. (20 pts) Consider the following heat equation problem:

$$\begin{aligned} u_t &= 4u_{xx} \\ \text{BCs}: \ u(0,t) &= 1, \ u(\pi, t) = 2 \\ \text{IC}: \ u(x,0) &= \frac{x}{\pi} \end{aligned}$$

(a) Using the PDE and the boundary conditions write the form of the solution $u(x,t)$.

(b) Now apply the initial condition to solve for the unknown coefficients in the solution from part (a).

Solution:

(a) From the PDE and the boundary conditions we have $L = \pi$, $\beta = 4$, $U_1 = 1$, and $U_2 = 2$. The solution to the PDE is of the form:

$$\boxed{u(x,t) = 1 + \frac{x}{\pi} + \sum_{n=1}^{\infty} c_n e^{-4n^2 t} \sin nx}$$

(b) Using the initial condition we have:

$$\begin{aligned} u(x,0) = 1 + \frac{x}{\pi} + \sum_{n=1}^{\infty} c_n \sin nx &= \frac{x}{\pi} \\ \sum_{n=1}^{\infty} c_n \sin nx &= -1 \end{aligned}$$

To solve for the c_n we must find the Sine Series for $g(x) = -1$:

$$\begin{aligned} c_n &= \frac{2}{\pi} \int_0^{\pi} (-1) \sin nx \, dx \\ c_n &= \frac{2}{\pi} \left[\frac{1}{n} \cos nx \right]_0^{\pi} \\ \Aboxed{c_n &= \frac{2}{n\pi} (\cos n\pi - 1)} \end{aligned}$$

Extra Credit: (10 pts) Find the solution to the following initial value problem (the wave equation for an "infinite" string):

$$\begin{aligned} u_{tt} &= u_{xx}, \quad -\infty < x < \infty \\ u(x,0) &= 1, \\ \frac{\partial u}{\partial t}(x,0) &= \cos x \end{aligned}$$

Hints:

1. The solution is of the form $u(x,t) = F(x + \alpha t) + G(x - \alpha t)$ where the functions F and G are to be determined from the initial conditions.
2. $\frac{\partial u}{\partial t}(x,t) = \alpha F'(x + \alpha t) - \alpha G'(x - \alpha t)$

Solution: We start by applying the initial conditions to the solution and to $\frac{\partial u}{\partial t}$ (computed in the second hint):

$$\begin{aligned} u(x,0) &= F(x) + G(x) = 1 \\ \frac{\partial u}{\partial t}(x,0) &= F'(x) - G'(x) = \cos x \end{aligned}$$

Integrating the second equation we get:

$$F(x) - G(x) = \sin x + C$$

Adding this equation to the first equation gives us:

$$2F(x) = 1 + \sin x + C \quad \Rightarrow \quad F(x) = \frac{1}{2} + \frac{1}{2}\sin x + \frac{C}{2}$$

We then solve for $G(x)$ using the first equation. We get:

$$G(x) = \frac{1}{2} - \frac{1}{2}\sin x - \frac{C}{2}$$

The solution is then:

$$\begin{aligned} u(x,t) &= F(x+t) + G(x-t) \\ u(x,t) &= \frac{1}{2} + \frac{1}{2}\sin(x+t) + \frac{C}{2} + \frac{1}{2} + \sin(x-t) - \frac{C}{2} \\ &\boxed{u(x,t) = 1 + \frac{1}{2}[\sin(x+t) - \sin(x-t)]} \end{aligned}$$