Math 417 – Midterm Exam Solutions
Friday, July 11, 2008

1. (30 pts) Determine all possible series representations of the function:

$$f(z) = \frac{1}{z(z^2+1)}$$

about $z = 0$ and state their regions of validity.

Solution: The function has singular points at $z = 0$, $z = i$, and $z = -i$. It will have a Laurent Series representation in the region $0 < |z| < 1$ and another Laurent Series representation in the region $1 < |z| < \infty$.

I. In the region $0 < |z| < 1$ we have

$$\begin{aligned} f(z) &= \frac{1}{z} \cdot \frac{1}{1+z^2} \\ &= \frac{1}{z}\left(1 - z^2 + (z^2)^2 - (z^2)^3 + \cdots\right) \\ &= \frac{1}{z} - z + z^3 - z^5 + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n z^{2n-1} \end{aligned}$$

II. In the region $1 < |z| < \infty$ we have

$$\begin{aligned} f(z) &= \frac{1}{z} \cdot \frac{1}{1+z^2} \\ &= \frac{1}{z} \cdot \frac{1}{z^2(1+\frac{1}{z^2})} \\ &= \frac{1}{z^3} \cdot \frac{1}{1+\frac{1}{z^2}} \\ &= \frac{1}{z^3}\left(1 - \frac{1}{z^2} + \left(\frac{1}{z^2}\right)^2 - \left(\frac{1}{z^2}\right)^3 + \cdots\right) \\ &= \frac{1}{z^3} - \frac{1}{z^5} + \frac{1}{z^7} - \frac{1}{z^9} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n+3}} \end{aligned}$$

2. (20 pts) Find all points at which

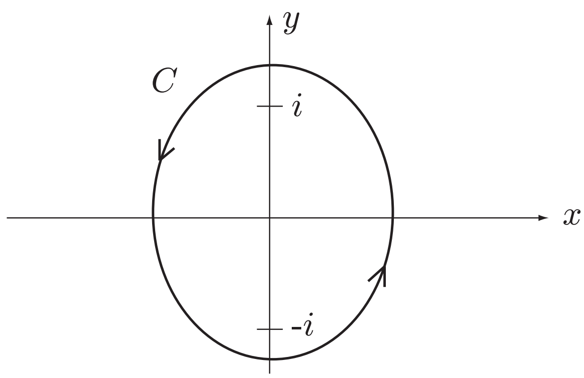
$$f(z) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

is differentiable. At what points is f analytic? Explain.

3. (40 pts) Compute each of the following integrals:

(a) $\int_C \sin\left(\frac{1}{z}\right)\, dz$ where C is the circle $|z| = 1$ oriented counterclockwise.

(b) $\int_C \frac{e^z}{z(z^2+1)}\, dz$ where the contour C is shown below:



Solution:

(a) The function $f(z) = \sin(\frac{1}{z})$ has a singular point at $z = 0$, which lies inside C. The Laurent Series of $f(z)$ around $z = 0$ in the region $0 < |z| < \infty$ is

$$f(z) = \sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \frac{1}{7!z^7} + \cdots$$

We can see that $z = 0$ is an essential singularity and that the residue at $z = 0$ is

$$\operatorname*{Res}_{z=0} f(z) = c_{-1} = 1$$

Using the Cauchy Residue Theorem, the value of the integral is

$$\int_C \sin\left(\frac{1}{z}\right)\, dz = 2\pi i \operatorname*{Res}_{z=0} f(z) = 2\pi i$$

(b) The function has singular points at $z = 0$, $z = i$, and $z = -i$.

I. Let $\phi_1(z) = \frac{e^z}{z^2+1}$. Clearly, $\phi_1(z)$ is analytic and nonzero at $z = 0$ and $f(z) = \frac{\phi_1(z)}{z}$. Therefore, $z = 0$ is a simple pole and the residue is

$$\operatorname*{Res}_{z=0} f(z) = \phi_1(0) = 1$$

II. Let $\phi_2(z) = \frac{e^z}{z(z+i)}$. Clearly, $\phi_2(z)$ is analytic and nonzero at $z = i$ and $f(z) = \frac{\phi_2(z)}{z-i}$. Therefore, $z = i$ is a simple pole and the residue is

$$\operatorname*{Res}_{z=i} f(z) = \phi_2(i) = -\frac{e^i}{2}$$

III. Let $\phi_3(z) = \frac{e^z}{z(z-i)}$. Clearly, $\phi_3(z)$ is analytic and nonzero at $z = -i$ and $f(z) = \frac{\phi_3(z)}{z+i}$. Therefore, $z = -i$ is a simple pole and the residue is

$$\operatorname*{Res}_{z=-i} f(z) = \phi_3(-i) = -\frac{e^{-i}}{2}$$

Using the Cauchy Residue Theorem, the value of the integral is

$$\begin{aligned}\int_C \frac{e^z}{z(z^2+1)}\,dz &= 2\pi i \left(\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=i} f(z) + \operatorname*{Res}_{z=-i} f(z) \right) \\ &= 2\pi i \left(1 - \frac{e^i}{2} - \frac{e^{-i}}{2} \right) \\ &= 2\pi i - \pi i e^i - \pi i e^{-i} \\ &= 2\pi i - \pi i(\cos 1 + i \sin 1) - \pi i(\cos 1 - i \sin 1) \\ &= 2\pi i(1 - \cos 1)\end{aligned}$$

4. (30 pts) Compute the improper integral:

$$I = \int_0^\infty \frac{dx}{x^2+x+1}$$

by considering the integral:

$$\int_C \frac{\log z}{z^2+z+1}\,dz$$

where C is the contour depicted in Figure 99 on p. 274.

5. (30 pts) Complete each of the following:

   (a) Find all values of z^π where $z = 2 + 2i$.

   (b) Determine the principal value of $\log z$ where $z = -1 - i$.

   (c) How many solutions of $3e^z - z = 0$ are in the disk $|z| \le 1$? Explain.

Solution:

(c) Let $f(z) = 3e^z$ and $g(z) = -z$. Both functions are analytic on and inside the circle $|z| = 1$. On the circle we have,

$$|f(z)| = |3e^z| = 3e^x > 3e^{-1} > 1$$

where the inequality is established by noting that e^x takes on its smallest value when x is most negative on the circle, which is at $x = -1$. We also have

$$|g(z)| = |-z| = |z| = 1$$

on the circle $|z| = 1$. Therefore, $|f(z)| > |g(z)|$ on the circle. Since $f(z)$ has no zeros inside the circle (it has no zeros anywhere), then $f(z)+g(z) = 3e^z - z$ has no zeros inside the circle.

6. (25 pts) Evaluate the improper integral:

$$\int_0^\infty \frac{x \sin ax}{x^2 + b^2}\, dx$$

Solution: To evaluate this integral consider the complex integral

$$\int_C \frac{ze^{iaz}}{z^2 + b^2}\, dz$$

where C is the contour shown below.

The integral over C can be split into the sum of the integrals over each part of the contour as follows:

$$\int_C \frac{ze^{iaz}}{z^2 + b^2}\, dz = \int_{C_1} \frac{ze^{iaz}}{z^2 + b^2}\, dz + \int_{C_R} \frac{ze^{iaz}}{z^2 + b^2}\, dz$$

Let's compute the value of each of the above three integrals in turn.

I. The integral over C can be solved using the Cauchy Residue Theorem.

$$\int_C \frac{ze^{iaz}}{z^2 + b^2}\, dz = 2\pi i \operatorname*{Res}_{z=bi} f(z)$$

where we only evaluate the residue at $z = bi$ because this is the only singular point of the integrand that lies inside C.

By letting $\phi(z) = \frac{ze^{iaz}}{z+bi}$, we have

$$f(z) = \frac{\phi(z)}{z - bi}$$

where ϕ is analytic and nonzero at $z = bi$. Therefore, $z = bi$ is a simple pole and the residue is

$$\operatorname*{Res}_{z=bi} f(z) = \phi(bi) = \frac{(bi)e^{ia(bi)}}{bi + bi} = \frac{1}{2}e^{-ab}$$

So the value of the integral over C is

$$\int_C \frac{ze^{iaz}}{z^2 + b^2}\, dz = 2\pi i \left(\frac{1}{2}e^{-ab}\right) = \pi i e^{-ab}$$

II. The integral over C_1 becomes

$$\int_{C_1} \frac{ze^{iaz}}{z^2 + b^2}\, dz = \int_{-R}^{R} \frac{xe^{iax}}{x^2 + b^2}\, dx$$

after parametrizing the contour using $z = x, \quad -R \leq x \leq R$.

III. Finally, we use Jordan's Lemma to show that the integral over C_R will tend to 0. We note that $f(z) = \frac{1}{z(z^2+1)}$ is analytic at all points in the upper half plane that are exterior to the circle, say, $|z| = 2$ and that

$$|f(z)| = \frac{1}{|z||z^2+1|} \leq \frac{1}{R(|z|^2-1)} \leq \frac{1}{R(R^2-1)}$$

for all points z on C_R. Then, since

$$\lim_{R\to\infty} \frac{1}{R(R^2-1)} = 0$$

we know that

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{iaz}\, dz = 0$$

Putting it all together we get

$$\begin{aligned}
\int_C \frac{ze^{iaz}}{z^2+b^2}\, dz &= \int_{C_1} \frac{ze^{iaz}}{z^2+b^2}\, dz + \int_{C_R} \frac{ze^{iaz}}{z^2+b^2}\, dz \\
\pi i e^{-ab} &= \text{P.V.} \int_{-\infty}^{\infty} \frac{xe^{iax}}{x^2+b^2}\, dx \\
\pi i e^{-ab} &= \text{P.V.} \int_{-\infty}^{\infty} \frac{x(\cos ax + i\sin ax)}{x^2+b^2}\, dx
\end{aligned}$$

as $R \to \infty$. Taking the imaginary parts of both sides we get

$$\pi e^{-ab} = \text{P.V.} \int_{-\infty}^{\infty} \frac{x\sin ax}{x^2+b^2}$$

Note that the integrand of the right hand side is an even function, so the principal value is the actual value of the integral. Furthermore,

$$\int_{-\infty}^{\infty} \frac{x\sin ax}{x^2+b^2}\, dx = 2\int_0^{\infty} \frac{x\sin ax}{x^2+b^2}\, dx$$

so that our final answer is

$$\int_0^{\infty} \frac{x\sin ax}{x^2+b^2}\, dx = \frac{\pi}{2}e^{-ab}$$