MATH 417 **Homework 2**
Instructor: D. Cabrera **Due June 23**

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1. Find and sketch the image of the rectangle $0 < x < 1$, $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$ under the transformation $w = e^z$.

   Solution: The transformation $w = e^z$ can be written as

   $$w = e^z = e^x \cos y + i e^x \sin y$$

   If we define $w = u + iv$ then the functions u and v are

   $$u = e^x \cos y, \quad v = e^x \sin y$$

   Let's consider the four boundaries:

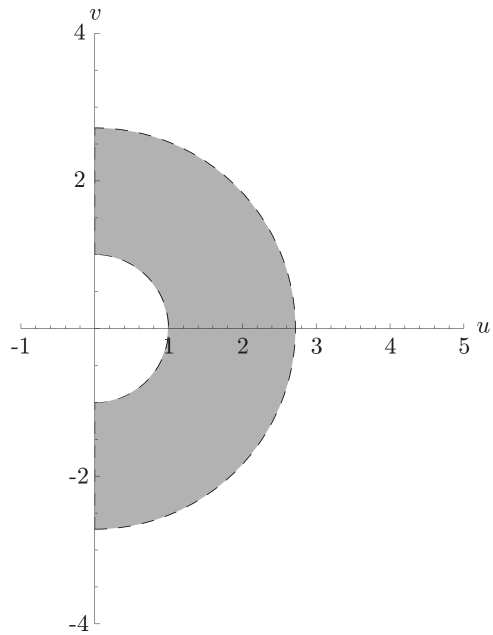
   i. The top of the rectangle is $y = \dfrac{\pi}{2}$, $0 \le x \le 1$. In this case, the functions u and v are
   $$u = e^x \cos \frac{\pi}{2} = 0, \quad v = e^x \sin \frac{\pi}{2} = e^x$$
   Thus, the image is the vertical line $u = 0$ where $e^0 \le v \le e^1$ or $1 \le v \le e$.

   ii. The right side of the rectangle is $x = 1$, $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$. In this case, the functions u and v are
   $$u = e^1 \cos y = e \cos y, \quad v = e^1 \sin y = e \sin y$$
   Thus, the image is the right half of the circle $u^2 + v^2 = e^2$.

   iii. The bottom of the rectangle is $y = -\dfrac{\pi}{2}$, $0 \le x \le 1$. In this case, the functions u and v are
   $$u = e^x \cos\left(-\frac{\pi}{2}\right) = 0, \quad v = e^x \sin\left(-\frac{\pi}{2}\right) = -e^x$$
   Thus, the image is the vertical line $u = 0$ where $-e^1 \le v \le -e^0$ or $-2 \le v \le -1$.

   iv. The left side of the rectangle is $x = 0$, $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$. In this case, the functions u and v are
   $$u = e^0 \cos y = \cos y, \quad v = e^0 \sin y = \sin y$$
   Thus, the image is the right half of the circle $u^2 + v^2 = 1$.

   Since the sides of the rectangle are not included in the set, their images are not included.

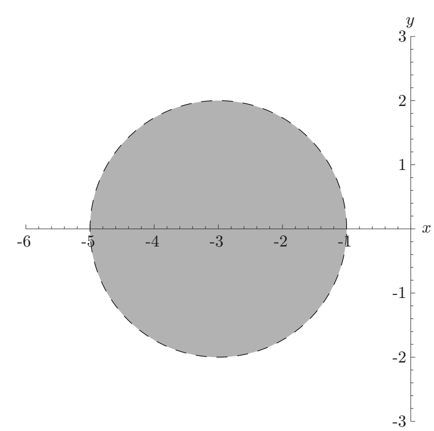


2. Sketch the following sets and determine whether they are open, closed, or neither.

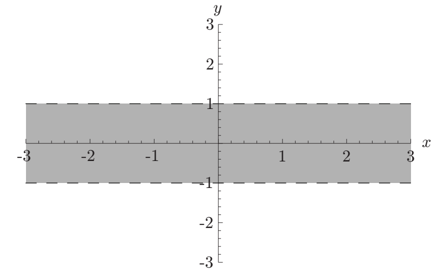
   (a) $|z+3| < 2$
   (b) $|\text{Im}\, z| < 1$
   (c) $0 < |z-1| < 2$
   (d) $\text{Re}\, z = 1$
   (e) $|z-4| \geq |z|$

Solution:

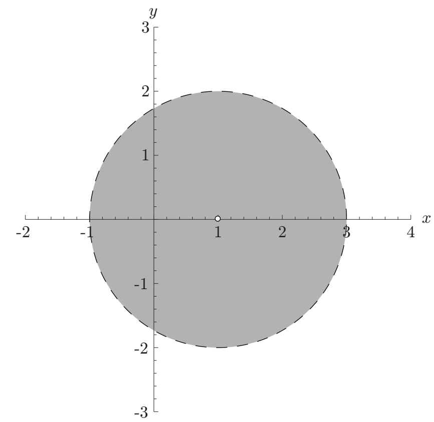
(a) The set $|z+3| < 2$ is the set of points inside the circle of radius 2 centered at $z = -3$. The set is open because it does not include any of its boundary points, which are all points on the circle $|z+3| = 2$.



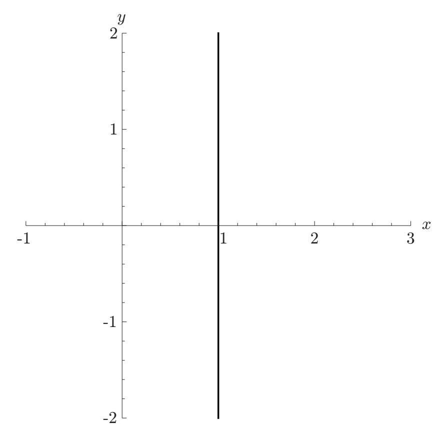
(b) The set $|\text{Im}\, z| < 1$ is the set of points satisfying $|y| < 1$ or $-1 < y < 1$. This is a vertical strip between the lines $y = -1$ and $y = 1$. The set is open because it does not include any of its boundary points, which are all points on the lines $y = -1$ and $y = 1$.



(c) The set $0 < |z-1| < 2$ is the set of points inside the circle of radius 2 centered at $z = 1$ except for the point $z = 1$. The set is open because it does not include any of its boundary points, which are all points on the circle $|z - 1| = 2$ and the point $z = 1$.



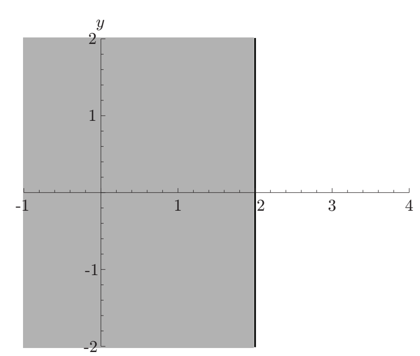
(d) The set $\text{Re}\, z = 1$ is the set of points satisfying $x = 1$. The set is closed because it contains all of its boundary points, which are all points in the set.



(e) The set $|z-4| \geq |z|$ can be better described by simplifying the inequality as follows:

$$
\begin{aligned}
|z - 4| &\geq |z| \\
|(x - 4) + iy| &\geq |x + iy| \\
\sqrt{(x - 4)^2 + y^2} &\geq \sqrt{x^2 + y^2} \\
(x - 4)^2 + y^2 &\geq x^2 + y^2 \\
x^2 - 8x + 16 + y^2 &\geq x^2 + y^2 \\
-8x + 16 &\geq 0 \\
x &\leq 2
\end{aligned}
$$

The set is open because it includes all of its boundary points, which are the points on the line $x = 2$.



3. Show that

(a) $\lim_{z\to 3} \frac{2}{z-3} = \infty$

(b) $\lim_{z\to\infty} \frac{z^2+1}{3z^2-4} = \frac{1}{3}$

Solution:

(a) To show that the limit is ∞, we use the fact that

$$\lim_{z\to z_0} f(z) = \infty \quad \Longleftrightarrow \quad \lim_{z\to z_0} \frac{1}{f(z)} = 0 \ .$$

Therefore, since

$$\lim_{z\to z_0} \frac{1}{f(z)} = \lim_{z\to 3} \frac{1}{\frac{2}{z-3}} = \lim_{z\to 3} \frac{z-3}{2} = 0$$

we know that

$$\lim_{z\to z_0} f(z) = \lim_{z\to 3} \frac{2}{z-3} = \infty \ .$$

(b) To show that the limit is $\frac{1}{3}$, we use the fact that

$$\lim_{z\to\infty} f(z) = w_0 \quad \Longleftrightarrow \quad \lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0 \ .$$

Therefore, since

$$\lim_{z\to 0} f\left(\frac{1}{z}\right) = \lim_{z\to 0} \frac{(\frac{1}{z})^2+1}{3(\frac{1}{z})^2-4} = \lim_{z\to 0} \frac{1+z^2}{3-4z^2} = \frac{1}{3}$$

we know that

$$\lim_{z\to\infty} f(z) = \lim_{z\to\infty} \frac{z^2+1}{3z^2-4} = \frac{1}{3} \ .$$

4. Find $f'(z)$ for the following functions:

(a) $f(z) = 4z^2 + 5z - 3$

(b) $f(z) = (2 - z^3)^2$

(c) $f(z) = \frac{z+2}{3z-2}$ where $z \neq \frac{2}{3}$

Solution: Using the derivative rules, we have

(a) $f'(z) = 8z + 5$

(b) $f'(z) = -6z^2(2 - z^3)$

(c) $f'(z) = \dfrac{(3z-2) - 3(z+2)}{(3z-2)^2} = \dfrac{-8}{(3z-2)^2}$

5. Show that $f'(z)$ does not exist at any point z when $f(z) = \operatorname{Re} z$.

   Solution: First, we have $f(z) = \operatorname{Re} z = \operatorname{Re}(x+iy) = x$. So $u(x,y) = x$ and $v(x,y) = 0$. These functions have continuous first derivatives everywhere in the complex plane. The first partial derivatives are

$$\begin{aligned} u_x &= 1, \quad v_y = 0 \\ u_y &= 0, \quad v_x = 0 \end{aligned}$$

   Clearly, the first Cauchy-Riemann equation $u_x = v_y$ is never satisfied as $1 \neq 0$. Therefore, the function $f(z)$ is not differentiable anywhere.

6. Let $z = x + iy$. Determine the values of z for which the Cauchy-Riemann equations are satisfied for the following functions:

   (a) $f(z) = e^{-x}e^{-iy}$
   (b) $f(z) = 2x + ixy^2$
   (c) $f(z) = x^2 + iy^2$
   (d) $f(z) = \operatorname{Im} z$

   Solution:

   (a) First we rewrite the given function as

$$f(z) = e^{-x}e^{-iy} = e^{-x}(\cos(-y) + i\sin(-y)) = e^{-x}\cos y - ie^{-x}\sin y$$

   Therefore, $u = e^{-x}\cos y$ and $v = -e^{-x}\sin y$. The first partial derivatives are

$$\begin{aligned} u_x &= -e^{-x}\cos y, \quad v_y = -e^{-x}\cos y \\ u_y &= -e^{-x}\sin y, \quad v_x = e^{-x}\sin y \end{aligned}$$

   We can see that the Cauchy-Riemann equations ($u_x = v_y,\ u_y = -v_x$) are satisfied for all $z = x + iy$.

   (b) Here we have $u = 2x$ and $v = xy^2$. The first partial derivatives are

$$\begin{aligned} u_x &= 2, \quad v_y = 2xy \\ u_y &= 0, \quad v_x = y^2 \end{aligned}$$

   In order for the second of the C-R equations to be satisfied, we must have

$$\begin{aligned} u_y &= -v_x \\ 0 &= -y^2 \\ y &= 0 \end{aligned}$$

However, if $y = 0$ then the first of the C-R equations gives us

$$\begin{aligned} u_x &= v_y \\ 2 &= 2xy \\ 2 &= 2x(0) \\ 2 &= 0 \end{aligned}$$

which is impossible. Therefore, the C-R equations are never satisfied.

(c) Here we have $u = x^2$ and $v = y^2$. The first partial derivatives are

$$\begin{aligned} u_x &= 2x, \quad v_y = 2y \\ u_y &= 0, \quad v_x = 0 \end{aligned}$$

The second of the C-R equations, $u_y = -v_x$, is always satisfied as $0 = 0$. In order for the first of the C-R equations to be satisfied we must have

$$\begin{aligned} u_x &= v_y \\ 2x &= 2y \\ x &= y \end{aligned}$$

Therefore, the C-R equations are satisfied for all z satisfying $z = x + ix$ where x is any real number.

(d) First we rewrite $f(z)$ as

$$f(z) = \operatorname{Im} z = \operatorname{Im}(x + iy) = y$$

Therefore we have $u = y$ and $v = 0$. The first partial derivatives are

$$\begin{aligned} u_x &= 0, \quad v_y = 0 \\ u_y &= 1, \quad v_x = 0 \end{aligned}$$

The second of the C-R equations is never satisfied because

$$\begin{aligned} u_y &= -v_x \\ 1 &= 0 \end{aligned}$$

is impossible. Therefore, the C-R equations are never satisfied.