MATH 417 **Homework 3**
Instructor: D. Cabrera **Due June 30**

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1. Let a function $f(z) = u + iv$ be differentiable at z_0.

   (a) Use the Chain Rule and the formulas $x = r\cos\theta$ and $y = r\sin\theta$ to show that

   $$u_x = u_r \cos\theta - u_\theta \frac{\sin\theta}{r}, \quad v_x = v_r \cos\theta - v_\theta \frac{\sin\theta}{r}$$

   (b) Then use the Cauchy-Riemann equations in polar coordinates

   $$ru_r = v_\theta, \quad u_\theta = -rv_r$$

   and the fact that $f'(z_0) = u_x + iv_x$ to show that

   $$f'(z_0) = e^{-i\theta}(u_r + iv_r)$$

Solution:

(a) Using the Chain Rule we have

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\ u_r &= u_x \cos\theta + u_y \sin\theta \qquad (1) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} \\ u_\theta &= u_x(-r\sin\theta) + u_y(r\cos\theta) \qquad (2) \end{aligned}$$

By multiplying Equation (1) by $r\cos\theta$ and Equation (2) by $\sin\theta$ and subtracting (2) from (1) we eliminate u_y and get

$$\begin{aligned} u_r r\cos\theta - u_\theta \sin\theta &= u_x r\cos^2\theta + u_x r\sin^2\theta \\ u_r r\cos\theta - u_\theta \sin\theta &= u_x r \\ \boxed{u_r \cos\theta - u_\theta \frac{\sin\theta}{r} = u_x} & \qquad (3) \end{aligned}$$

which is what we wanted to show. By replacing u with v we get the other equation

$$\boxed{v_r \cos\theta - v_\theta \frac{\sin\theta}{r} = v_x} \qquad (4)$$

(b) We will now use the Cauchy-Riemann equations to replace the u_θ term in Equation (3) with $-rv_r$ and the v_θ term in Equation (4) with ru_r and get

$$\begin{aligned} u_x &= u_r \cos\theta - (-rv_r)\frac{\sin\theta}{r} \\ &= u_r \cos\theta + v_r \sin\theta \\ v_x &= v_r \cos\theta - ru_r\frac{\sin\theta}{r} \\ &= v_r \cos\theta - u_r \sin\theta \end{aligned}$$

Finally, we plug these expressions for u_x and v_x into the derivative $f'(z) = u_x + iv_x$ and simplify to get $f'(z)$ in polar coordinates.

$$\begin{aligned} f'(z) &= u_x + iv_x \\ &= u_r \cos\theta + v_r \sin\theta + i(v_r \cos\theta - u_r \sin\theta) \\ &= u_r(\cos\theta - i\sin\theta) + v_r(\sin\theta + i\cos\theta) \\ &= u_r(\cos\theta - i\sin\theta) + iv_r(\cos\theta - i\sin\theta) \\ &= u_r(\cos(-\theta) + i\sin(-\theta)) + iv_r(\cos(-\theta) + i\sin(-\theta)) \\ &= u_r e^{-i\theta} + iv_r e^{-i\theta} \end{aligned}$$

$$\boxed{f'(z) = e^{-i\theta}(u_r + iv_r)}$$

2. Show that the function $f(z) = e^{-y}\sin x - ie^{-y}\cos x$ is entire.

Solution: Let $u(x, y) = e^{-y}\sin x$ and $v(x, y) = -e^{-y}\cos x$. We can see that u and v have continuous derivatives of all orders everywhere in the complex plane. Furthermore, the first partial derivatives of u and v are

$$\begin{aligned} u_x &= e^{-y}\cos x, \quad v_y = e^{-y}\cos x \\ u_y &= -e^{-y}\sin x, \quad v_x = e^{-y}\sin x \end{aligned}$$

so we can see that the Cauchy-Riemann equations ($u_x = v_y,\ u_y = -v_x$) are satisfied for all x,y. Therefore, $f'(z)$ exists for all z in the complex plane and $f(z)$ is entire.

3. Show that the function $f(z) = xy + iy$ is not analytic at any point in the complex plane.

Solution: Let $u(x, y) = xy$ and $v(x, y) = y$. The functions have continuous partial derivatives of all orders everywhere in the complex plane and the first partial derivatives are

$$\begin{aligned} u_x &= y, \quad v_y = 1 \\ u_y &= x, \quad v_x = 0 \end{aligned}$$

The Cauchy-Riemann equations ($u_x = v_y,\ u_y = -v_x$) are only satisfied when $y = 1$ and $x = 0$. Recall that a function is analytic at a point z_0 if it is analytic at every point in some neighborhood of z_0. Since $f'(z)$ exists only at $z = i$, there is no neighborhood of $z = i$ which has the property that $f'(z)$ exists at every point in that neighborhood. Thus, $f(z)$ is analytic nowhere.

4. Let $u(x, y) = \dfrac{y}{x^2 + y^2}$.

   (a) Show that $u(x, y)$ is harmonic in the domain D which is the set of all points z in the complex plane excluding $z = 0$.

   (b) Find the most general harmonic conjugate v of u.

Solution:

(a) First, we note that $u(x, y)$ has continuous first and second derivatives at every point in D. Now we must show that $u_{xx} + u_{yy} = 0$. The first and second partial derivatives are

$$u_x = -\frac{2xy}{(x^2+y^2)^2}, \quad u_{xx} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$$
$$u_y = \frac{x^2 - y^2}{(x^2+y^2)^2}, \quad u_{yy} = \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}$$

Clearly, the derivatives u_{xx} and u_{yy} add up to 0. Therefore, $u(x, y)$ is harmonic in D.

(b) A harmonic conjugate $v(x, y)$ of $u(x, y)$ must satisfy the Cauchy-Riemann equations. So we must have

$$v_y = u_x = -\frac{2xy}{(x^2+y^2)^2}$$
$$v_x = -u_y = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

Integrating the first of the above equations with respect to y we have

$$v_y = -\frac{2xy}{(x^2+y^2)^2}$$
$$\int v_y \, dy = -\int \frac{2xy}{(x^2+y^2)^2} \, dy$$
$$v(x, y) = -\int \frac{2xy}{(x^2+y^2)^2} \, dy$$

To evaluate the integral we let $w = x^2 + y^2$, $dw = 2y\,dy$.

$$v(x,y) = -\int \frac{x}{w^2}\,dw$$
$$v(x,y) = \frac{x}{w} + \phi(x)$$
$$v(x,y) = \frac{x}{x^2+y^2} + \phi(x)$$

To find the function $\phi(x)$ we differentiate the above expression for $v(x,y)$ with respect to x to get

$$\frac{\partial}{\partial x} v(x,y) = \frac{\partial}{\partial x}\frac{x}{x^2+y^2} + \frac{\partial}{\partial x}\phi(x)$$
$$v_x = \frac{y^2-x^2}{(x^2+y^2)^2} + \phi'(x)$$

The second of the Cauchy-Riemann equations tells us that

$$v_x = \frac{y^2-x^2}{(x^2+y^2)^2}$$

so it must be the case that $\phi'(x) = 0$, i.e. $\phi(x) = C =$ constant. Therefore, the most general harmonic conjugate of $u(x,y)$ is

$$\boxed{v(x,y) = \frac{x}{x^2+y^2} + C}$$

5. Find all values of each expression.

   (a) $\exp\left(2 - \frac{\pi}{4}i\right)$

   (b) $\log(-2+2i)$

   (c) $\operatorname{Log}(ei)$

Solution:

(a) $\exp\left(2 - \frac{\pi}{4}i\right) = e^2\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right] = \boxed{e^2\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)}$

(b) The modulus and principal argument of $z = -2+2i$ are

$$r = 2\sqrt{2}, \quad \Theta = \frac{3\pi}{4}$$

The logarithm of z is then

$$\log z = \ln r + i\,(\Theta + 2k\pi)$$

$$\boxed{\log(-2+2i) = \ln\left(2\sqrt{2}\right) + i\left(\frac{3\pi}{4} + 2k\pi\right)}$$

where $k = 0, \pm 1, \pm 2, \ldots$.

(c) The modulus and principal argument of $z = ei$ are

$$r = e, \quad \Theta = \frac{\pi}{2}$$

The principal logarithm of z is then

$$\operatorname{Log} z = \ln r + i\Theta$$
$$\operatorname{Log}\,(ei) = \ln e + i\frac{\pi}{2}$$
$$\boxed{\operatorname{Log}\,(ei) = 1 + i\frac{\pi}{2}}$$

6. Show that the function $f(z) = e^{2z}$ is entire and write an expression for $f'(z)$ in terms of z.

Solution: It is enough to say that $g(z) = 2z$ and $h(z) = e^z$ are entire so their composite $f(z) = h(g(z)) = e^{2z}$ is also entire. Using the Chain Rule, the derivative $f'(z)$ is

$$f'(z) = 2e^{2z}$$

We can also solve the problem by writing $f(z)$ in terms of x and y.

$$f(z) = e^{2(x+iy)} = e^{2x}\cos 2y + ie^{2x}\sin 2y$$

Note that the functions $u(x,y) = e^{2x}\cos 2y$ and $v(x,y) = e^{2x}\sin 2y$ have continuous first derivatives everywhere in the complex plane. Also, the Cauchy-Riemann equations are satisfied for all x, y as

$$u_x = v_y = 2e^{2x}\cos 2y$$
$$u_y = -v_x = -2e^{2x}\sin 2y$$

Therefore, $f(z) = e^{2z}$ is entire. The derivative $f'(z)$ is

$$f'(z) = u_x + iv_x$$
$$f'(z) = 2e^{2x}\cos 2y + i(2e^{2x}\sin 2y)$$
$$f'(z) = 2e^{2x}e^{i(2y)}$$
$$f'(z) = 2e^{2(x+iy)}$$
$$f'(z) = 2e^{2z}$$

7. Show that $\operatorname{Log}(-1+i)^2 \neq 2\operatorname{Log}(-1+i)$.

Solution: First, we evaluate the left hand side. The number $(-1+i)^2$ can be rewritten as $-2i$. The modulus and principal argument of $-2i$ are $r = 2$ and $\Theta = -\dfrac{\pi}{2}$, respectively. Therefore, the principal logarithm of $(-1+i)^2$ is

$$\operatorname{Log}(-1+i)^2 = \ln 2 - i\frac{\pi}{2}$$

Next, we evaluate the right hand side. The modulus and principal argument of $-1+i$ are $r = \sqrt{2}$ and $\Theta = \dfrac{3\pi}{4}$, respectively. Therefore, twice the principal logarithm of $-1+i$ is

$$2\operatorname{Log}(-1+i) = 2\left(\ln\sqrt{2} + i\frac{3\pi}{4}\right) = \ln 2 + i\frac{3\pi}{2}$$

Clearly, $\operatorname{Log}(-1+i)^2 \neq 2\operatorname{Log}(-1+i)$.