MATH 417 **Homework 4**
Instructor: D. Cabrera **Due July 7**

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1. Find all values of each expression below.

   (a) $(1-i)^i$

   (b) $\cos(1-i)$

   (c) $\sin^{-1}(2)$

   Solution:

   (a) Here we use the formula

$$z^c = e^{c \log z}$$
$$(1-i)^i = e^{i \log(1-i)}$$

The modulus of $1-i$ is $r = \sqrt{2}$ and the principal argument is $\Theta = -\dfrac{\pi}{4}$. Therefore,

$$\log(1-i) = \ln\sqrt{2} + i\left(-\frac{\pi}{4} + 2k\pi\right), \quad k = 0, \pm 1, \pm 2, \dots$$

Multiplying $\log(1-i)$ by i we get

$$i\log(1-i) = i\left[\ln\sqrt{2} + i\left(-\frac{\pi}{4} + 2k\pi\right)\right]$$
$$i\log(1-i) = \left(\frac{\pi}{4} + 2k\pi\right) + i\left(\ln\sqrt{2}\right)$$

Finally, we exponentiate $i\log(1-i)$ to get

$$(1-i)^i = e^{\pi/4 + 2k\pi + i\ln\sqrt{2}}$$
$$(1-i)^i = e^{\pi/2 + 2k\pi} e^{i\ln\sqrt{2}}$$
$$\boxed{(1-i)^i = e^{\pi/4 + 2k\pi}\left[\cos\left(\ln\sqrt{2}\right) + i\sin\left(\ln\sqrt{2}\right)\right]}$$

where $k = 0, \pm 1, \pm 2, \dots$.

   (b) Here we can use either

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$
$$\cos(1-i) = \frac{e^{i(1-i)} + e^{-i(1-i)}}{2}$$
$$\boxed{\cos(1-i) = \frac{e^{1+i} + e^{-1-i}}{2}}$$

or

$$\cos z = \cosh x \cos y - i \sinh x \sin y$$
$$\cos(1-i) = \cosh 1 \cos(-1) - i \sinh 1 \sin(-1)$$
$$\boxed{\cos(1-i) = \cosh 1 \cos 1 + i \sinh 1 \sin 1}$$

(c) Here we use the formula

$$\sin^{-1} z = -i \log \left[iz + (1-z^2)^{1/2}\right]$$
$$\sin^{-1} 2 = -i \log \left[2i \pm \sqrt{3}i\right]$$
$$\sin^{-1} 2 = -i \log \left[(2 \pm \sqrt{3})i\right]$$

If we take the positive root, then we have

$$\sin^{-1} 2 = -i \log \left[(2+\sqrt{3})i\right]$$
$$\sin^{-1} 2 = -i \left[\ln(2+\sqrt{3}) + i\left(\frac{\pi}{2} + 2k\pi\right)\right]$$
$$\boxed{\sin^{-1} 2 = \frac{\pi}{2} + 2k\pi - i\ln(2+\sqrt{3})}$$

If we take the negative root, then we have

$$\sin^{-1} 2 = -i \log \left[(2-\sqrt{3})i\right]$$
$$\sin^{-1} 2 = -i \left[\ln(2-\sqrt{3}) + i\left(\frac{\pi}{2} + 2k\pi\right)\right]$$
$$\boxed{\sin^{-1} 2 = \frac{\pi}{2} + 2k\pi - i\ln(2-\sqrt{3})}$$

where $k = 0, \pm 1, \pm 2, \ldots$.

2. Prove that $\sin(2z) = 2\sin z \cos z$ by using the definitions of $\sin z$ and $\cos z$.

Solution: Using the definition of $\sin z$ we have

$$\sin(2z) = \frac{e^{i(2z)} - e^{-i(2z)}}{2i}$$
$$\sin(2z) = \frac{(e^{iz} - e^{-iz})(e^{iz} + e^{-iz})}{2i}$$
$$\sin(2z) = 2\left(\frac{e^{iz} - e^{-iz}}{2i}\right)\left(\frac{e^{iz} + e^{-iz}}{2}\right)$$
$$\sin(2z) = 2\sin z \cos z$$

where in the last step, we used the definitions of $\sin z$ and $\cos z$.

3. Find the values of z for which $\cos z = 0$ by using the fact that

$$|\cos z|^2 = \cos^2 x + \sinh^2 y \qquad \text{where} \qquad \sinh y = \frac{e^y - e^{-y}}{2}$$

Solution: If $\cos z = 0$ then $|\cos z| = 0$. So it must be the case that both

$$\cos x = 0 \quad \text{and} \quad \sinh y = 0$$

happen simultaneously. From the first equation we have

$$x = \frac{(2k+1)\pi}{2}, \qquad k = 0, \pm 1, \pm 2, \dots$$

From the second equation we have $y = 0$. Therefore, $\cos z = 0$ when

$$\boxed{z = \frac{(2k+1)\pi}{2}, \qquad k = 0, \pm 1, \pm 2, \dots}$$

4. Show that $f(z) = \sin(\bar{z})$ is analytic nowhere.

Solution: The function can be written as

$$\begin{aligned}\sin(\bar{z}) &= \sin(x - iy)\\ \sin(\bar{z}) &= \sin x \cosh(-y) + i \cos x \sinh(-y)\\ \sin(\bar{z}) &= \sin x \cosh y - i \cos x \sinh y\end{aligned}$$

Letting $u = \sin x \cosh y$ and $v = -\cos x \sinh y$ and computing their first partial derivatives we get

$$\begin{aligned}u_x &= \cos x \cosh y, \quad v_y = -\cos x \cosh y\\ u_y &= \sin x \sinh y, \quad v_x = \sin x \sinh y\end{aligned}$$

In order for the Cauchy-Riemann equations ($u_x = v_y$, $u_y = -v_x$) to be satisfied, we need

$$\cos x \cosh y = 0 \quad \text{and} \quad \sin x \sinh y = 0$$

to occur simultaneously. From the first equation we can only have $\cos x = 0$ since $\cosh y > 0$ for all y. Therefore,

$$x = \frac{(2k+1)\pi}{2}, \quad k = 0, \pm 1, \pm 2, \dots$$

From the second equation we must have $\sinh y = 0$ because $\sin x$ and $\cos x$ cannot be 0 simultaneously. Therefore, $y = 0$.

Thus, since the first partial derivatives of u and v are continuous everywhere in the complex plane and the Cauchy-Riemann equations are satisfied for $z = \dfrac{(2k+1)\pi}{2}$, $f'(z)$ exists for these values of z. However, at each point there is no neighborhood throughout which $f(z)$ is analytic. Therefore, $f(z) = \sin(\bar{z})$ is analytic nowhere.

5. Evaluate the integral

$$\int_C e^z \, dz$$

where C is the contour consisting of the two straight-line segments: (1) from $z = i$ to $z = 1 + i$ and (2) from $z = 1 + i$ to $z = 1 - 2i$.

Solution: To evaluate the integral we integrate over each line segment and then add the results. On the first segment we have the parametrization

$$z(t) = t + i, \quad 0 \le t \le 1$$

Therefore, the integral of $f(z)$ over this segment is

$$\begin{aligned} \int_{C_1} f(z) \, dz &= \int_0^1 f(z(t)) z'(t) \, dt \\ &= \int_0^1 e^{t+i} (1) \, dt \\ &= e^{t+i} \Big|_0^1 \\ &= e^{1+i} - e^i \end{aligned}$$

On the second segment we have the parametrization

$$z(t) = 1 + it, \quad -2 \le t \le 1$$

Therefore, the integral of $f(z)$ over this segment is

$$\begin{aligned} \int_{C_2} f(z) \, dz &= \int_1^{-2} f(z(t)) z'(t) \, dt \\ &= \int_1^{-2} e^{1+it} (i) \, dt \\ &= e^{1+it} \Big|_1^{-2} \\ &= e^{1-2i} - e^{1+i} \end{aligned}$$

The value of the integral is then

$$\int_C f(z) \, dz = \int_{C_1} f(z) \, dz + \int_{C_2} f(z) \, dz$$

$$\begin{aligned} \int_C f(z) \, dz &= e^{1+i} - e^i + e^{1-2i} - e^{1+i} \\ &= e^{1-2i} - e^i \qquad = e(\cos 2 - i \sin 2) + e(\cos 1 + i \sin 1) \\ &= \boxed{e \left[(\cos 2 + \cos 1) + i \left(\sin 1 - \sin 2 \right) \right]} \end{aligned}$$

Note: Instead of using parametrizations, we could have said that $f(z)$ is entire so it has an antiderivative $F(z) = e^z$ and the value of the integral is

$$\int_C e^z \, dz = F(1-2i) - F(i) = e^{1-2i} - e^i$$

which is exactly what we obtained above.

6. Evaluate the integral

$$\int_C (z^2 - 1) \, dz$$

where C is the semicircle $z = e^{it}$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$ oriented counterclockwise.

Solution: The value of the integral is

$$\begin{aligned}
\int_C f(z) \, dz &= \int_a^b f(z(t))z'(t) \, dt \\
\int_C (z^2 - 1) \, dz &= \int_{-\pi/2}^{\pi/2} \left(e^{2it} - 1\right) ie^{it} \, dt \\
&= i \int_{-\pi/2}^{\pi/2} \left(e^{3it} - e^{it}\right) \, dt \\
&= i \left[\frac{1}{3i}e^{3it} - \frac{1}{i}e^{it}\right]_{-\pi/2}^{\pi/2} \\
&= \left(\frac{1}{3}e^{i(3\pi/2)} - e^{i(\pi/2)}\right) - \left(\frac{1}{3}e^{i(-3\pi/2)} - e^{i(-\pi/2)}\right) \\
&= -\frac{1}{3}i - i - \frac{1}{3}i - i \\
&= \boxed{-\frac{8}{3}i}
\end{aligned}$$

Note: Instead of using the parametrization, we could have said that $f(z)$ is entire so it has an antiderivative $F(z) = \frac{1}{3}z^3 - z$ and the value of the integral is

$$\int_C (z^2-1) \, dz = F(i)-F(-i) = \left(\frac{1}{3}i^3 - i\right) - \left(\frac{1}{3}(-i)^3 - (-i)\right) = -\frac{1}{3}i - i - \frac{1}{3}i - i = -\frac{8}{3}i$$

which is exactly what we obtained above.

7. Show that

$$\left| \int_C \frac{2z+1}{z^2-4} \, dz \right| \le \pi$$

where C is the upper half of the circle $|z| = 1$ oriented counterclockwise. Justify your answer.

Solution: The length of the contour is $L = \pi$. Now we must find an upper bound on $|f(z)|$. Using the triangle inequality $|z_1 + z_2| \leq |z_1| + |z_2|$ on the numerator we have

$$|2z + 1| \leq 2|z| + 1 = 2 + 1 = 3$$

Using the triangle inequality $|z_1 - z_2| \geq ||z_1| - |z_2||$ on the denominator we have

$$|z^2 - 4| \geq ||z|^2 - 4| = |1 - 4| = 3$$

Thus, the modulus of $f(z)$ satisfies the inequality

$$|f(z)| = \left|\frac{2z+1}{z^2-4}\right| \leq \frac{3}{3} = 1$$

Choosing $M = 1$ and using the formula for the ML-Bound we have

$$\left|\int_C \frac{2z+1}{z^2-4}\, dz\right| \leq ML = \pi$$

8. Find an upper bound on

$$\left|\int_C \frac{dz}{z^2+1}\right|$$

where C is the circle $|z - i| = 1$ oriented counterclockwise. Justify your answer.

Solution: The length of the contour is $L = 2\pi$. To find an upper bound on $|f(z)|$ we'll factor $z^2 + 1$ and take the modulus to get

$$\left|\frac{1}{z^2+1}\right| = \frac{1}{|z-i||z+i|} = \frac{1}{1 \cdot |(z-i) + 2i|} = \frac{1}{|(z-i) + 2i|}$$

Now we use the triangle inequality $|z_1 + z_2| \geq ||z_1| - |z_2||$ on the denominator to get

$$|(z-i) + 2i| \geq ||z - i| - |2i|| = |1 - 2| = 1$$

Thus, we have

$$\left|\frac{1}{z^2+1}\right| \leq \frac{1}{1} = 1$$

Choosing $M = 1$ and using the ML-Bound formula we have

$$\left|\int_C \frac{dz}{z^2+1}\right| \leq ML = 2\pi$$