MATH 417 **Homework 5**
Instructor: D. Cabrera **Due July 16**

1. Show that
$$\int_C f(z)\,dz = 0$$
where C is the circle $|z| = 2$ oriented clockwise for each function below:

   (a) $f(z) = ze^{-z}$

   (b) $f(z) = \dfrac{1}{z^2+9}$

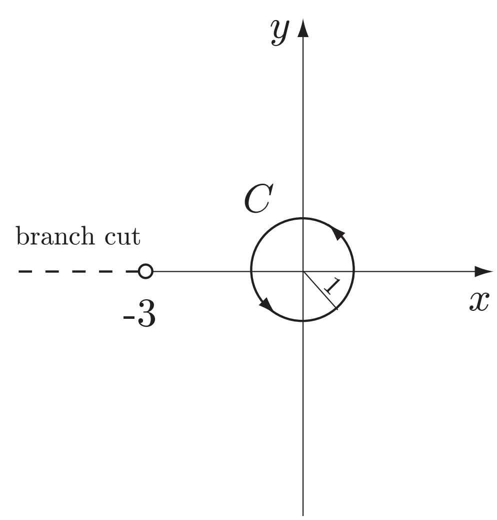
   Solution:

   (a) The function $f(z) = ze^{-z}$ is entire and C is a simple closed contour. By the Cauchy-Goursat Theorem, the value of the integral is 0.

   (b) The function $f(z) = \dfrac{1}{z^2+9}$ has singular points at $x = 3i$ and $x = -3i$. Each of these points lies outside of C. Therefore, $f(z)$ is analytic everywhere on and inside C so, by the Cauchy-Goursat Theorem, the value of the integral is 0.

2. If C is the unit circle $|z| = 1$ oriented clockwise, then is
$$\int_C \operatorname{Log}(z+3)\,dz = 0\ ?$$
Why or why not? Recall that $\operatorname{Log} z$ is the principal logarithm where $|z| > 0$ and $-\pi < \arg z < \pi$.

   Solution: The function $\operatorname{Log}(z+3)$ has singular points at all points on the branch cut which starts at $z = -3$ and extends along the negative real axis. These points lie outside of the unit circle so, by the Cauchy-Goursat Theorem, the value of the integral is 0.



3. Evaluate

$$\int_C \frac{dz}{z^2-1}$$

where C is the circle $|z| = 2$ oriented counterclockwise.

Solution: The function $\dfrac{1}{z^2-1} = \dfrac{1}{(z+1)(z-1)}$ has singularities at $z = -1, 1$ and both points lie inside the contour C. We can evaluate the integral in a few different ways. One way is to start with a partial fraction decomposition of the function:

$$\frac{1}{(z+1)(z-1)} = \frac{-\frac{1}{2}}{z+1} + \frac{\frac{1}{2}}{z-1}$$

Then we have

$$\int_C \frac{dz}{z^2-1} = -\frac{1}{2}\int_C \frac{dz}{z+1} + \frac{1}{2}\int_C \frac{dz}{z-1}$$

Now we can evaluate each integral using the Cauchy Integral Formula as C is a simple closed contour:

$$\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$$

In the first integral, we have $f(z) = 1$, which is analytic on and inside C, and $z_0 = -1$ so

$$\int_C \frac{dz}{z+1} = 2\pi i(1) = 2\pi i$$

In the second integral, we have $f(z) = 1$, which is analytic on and inside C, and $z_0 = 1$ so

$$\int_C \frac{dz}{z-1} = 2\pi i(1) = 2\pi i$$

The value of the integral is then:

$$\boxed{\int_C \frac{dz}{z^2-1} = -\frac{1}{2}(2\pi i) + \frac{1}{2}(2\pi i) = 0}$$

4. Evaluate

$$\int_C \frac{\cos z}{z(z+2)}\,dz$$

where C is the square of side 6 centered at $z = 0$ and oriented counterclockwise.

Solution: The function $\dfrac{\cos z}{z(z+2)}$ has singularities at $z = -2, 0$ and both points lie inside the contour C. Again, we can evaluate the integral in a few different ways. One way is to start with a partial fraction decomposition of the function:

$$\frac{\cos z}{z(z+2)} = \frac{1}{2} \cdot \frac{\cos z}{z} - \frac{1}{2} \cdot \frac{\cos z}{z+2}$$

Then we have

$$\int_C \frac{\cos z}{z(z+2)} = \frac{1}{2} \int_C \frac{\cos z}{z}\, dz - \frac{1}{2} \int_C \frac{\cos z}{z+2}\, dz$$

Now we can evaluate each integral using the Cauchy Integral Formula as C is a simple closed contour. In the first integral, we have $f(z) = \cos z$, which is analytic on and inside C, and $z_0 = 0$ so

$$\int_C \frac{\cos z}{z}\, dz = 2\pi i \cos 0 = 2\pi i$$

In the second integral, we have $f(z) = \cos z$, which is analytic on and inside C, and $z_0 = -2$ so

$$\int_C \frac{\cos z}{z+2}\, dz = 2\pi i \cos(-2) = 2\pi i \cos 2$$

The value of the integral is then:

$$\boxed{\int_C \frac{\cos z}{z(z+2)}\, dz = \frac{1}{2}(2\pi i) - \frac{1}{2}(2\pi i \cos 2) = \pi i(1 - \cos 2)}$$

5. Evaluate

$$\int_C \frac{e^z}{(z-\pi)^3}\, dz$$

where C is the square of side 8 centered at $z = 0$ oriented counterclockwise.

Solution: First, we recognize that $z = \pi$ is a singular point of the integrand and lies inside the simple closed contour C. Now let $f(z) = e^z$ which is analytic on and inside C. We can then use the extended Cauchy Integral Formula:

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

where $n = 2$ and $z_0 = \pi$. The value of the integral is then:

$$\boxed{\int_C \frac{e^z}{(z-\pi)^3}\, dz = \frac{2\pi i}{2!} \frac{d^2}{dz^2} e^z \Big|_{z=\pi} = \pi i e^\pi}$$

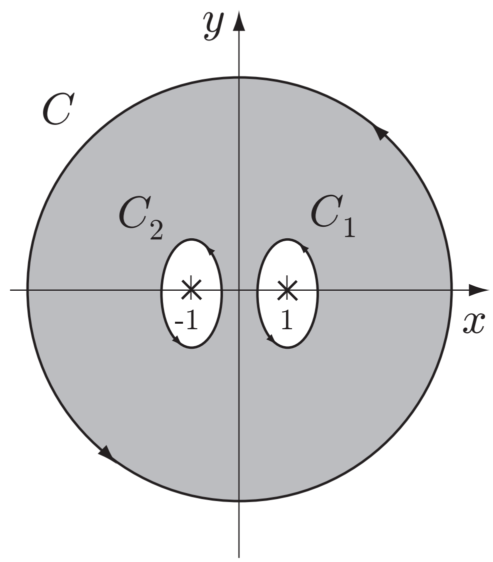
6. Evaluate

$$\int_C \frac{2z+1}{z^4-2z^2+1}\,dz$$

where C is the circle $|z| = 10$ oriented clockwise.

Solution: First, we'll factor the integrand into:

$$\frac{2z+1}{z^4-2z^2+1} = \frac{2z+1}{(z-1)^2(z+1)^2}$$



The singular points of the function are $z = 1, -1$. Instead of performing the partial fraction decomposition, we will deform C into two simple closed contours oriented counterclockwise, each of which encloses a singular point. This is possible because the function is analytic on C, the new contours, and everywhere in between (the region in gray). Let C_1 be the contour enclosing $z = 1$ and C_2 be the contour enclosing $z = -1$. Then

$$\int_C \frac{2z+1}{z^4-2z^2+1}\,dz = \int_{C_1} \frac{\frac{2z+1}{(z+1)^2}}{(z-1)^2}\,dz + \int_{C_2} \frac{\frac{2z+1}{(z-1)^2}}{(z+1)^2}\,dz$$

In the first integral, we have $f(z) = \dfrac{2z+1}{(z+1)^2}$ which is analytic everywhere on and inside C_1, $z_0 = 1$, and $n = 1$. Using the extended Cauchy Integral Formula we have:

$$\int_{C_1} \frac{\frac{2z+1}{(z+1)^2}}{(z-1)^2}\,dz = \frac{2\pi i}{1!}\frac{d}{dz}\left(\frac{2z+1}{(z+1)^2}\right)\Bigg|_{z=1} = 2\pi i\left(-\frac{1}{4}\right) = -\frac{\pi i}{2}$$

In the second integral, we have $f(z) = \dfrac{2z+1}{(z-1)^2}$ which is analytic everywhere on and inside C_2, $z_0 = -1$, and $n = 1$. Using the extended Cauchy Integral Formula we have:

$$\int_{C_2} \frac{\frac{2z+1}{(z-1)^2}}{(z+1)^2}\,dz = \frac{2\pi i}{1!}\frac{d}{dz}\left(\frac{2z+1}{(z-1)^2}\right)\Bigg|_{z=-1} = 2\pi i\left(\frac{1}{4}\right) = \frac{\pi i}{2}$$

Thus, the value of the integral is

$$\boxed{\int_C \frac{2z+1}{z^4-2z^2+1}\,dz = -\frac{\pi i}{2} + \frac{\pi i}{2} = 0}$$