MATH 417 **Homework 6**
Instructor: D. Cabrera **Due July 21**

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1. Find the radius of convergence for each power series below.

   (a) $\displaystyle\sum_{n=2}^{\infty} n^2(z-3)^n$

   (b) $\displaystyle\sum_{n=4}^{\infty} e^n(z+i)^n$

Solution:

(a) Using the Ratio Test we have

$$\begin{aligned} L &= \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(n+1)^2(z-3)^{n+1}}{n^2(z-3)^n}\right| \\ &= \lim_{n\to\infty}\frac{(n+1)^2}{n^2}|z-3| \\ &= |z-3|\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 \\ &= |z-3| \end{aligned}$$

The series converges when $L = |z-3| < 1$. Therefore, the radius of convergence is 1.

(b) Using the Ratio Test we have

$$\begin{aligned} L &= \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| \\ &= \lim_{n\to\infty}\left|\frac{e^{n+1}(z+i)^{n+1}}{e^n(z+i)^n}\right| \\ &= \lim_{n\to\infty} e|z+i| \\ &= e|z+i| \end{aligned}$$

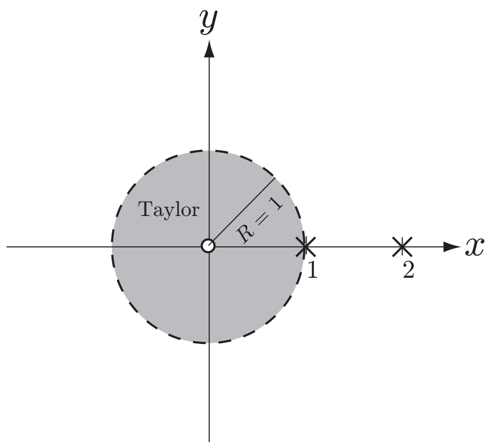
The series converges when $L = e|z+i| < 1 \implies |z+i| < \dfrac{1}{e}$. Therefore, the radius of convergence is $\dfrac{1}{e}$.

2. What is the radius of convergence of the Taylor Series of $f(z) = \dfrac{1}{z^2 - 3z + 2}$ about $z = 0$? about $z = 3i$?

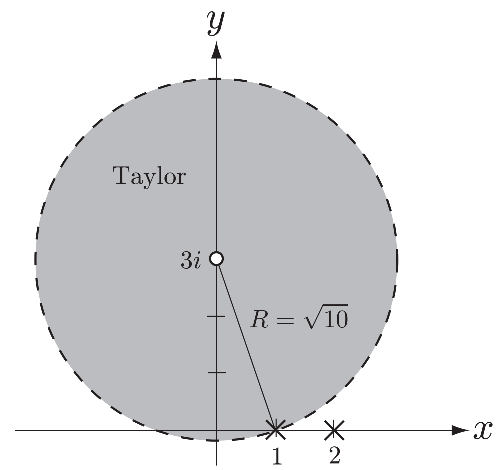
Solution: The singular points of $f(z) = \dfrac{1}{z^2 - 3z + 2} = \dfrac{1}{(z-1)(z-2)}$ are $z = 1$ and $z = 2$. Therefore, since $f(z)$ is analytic at $z = 0$, it has a Taylor Series representation for all z satisfying $|z| < R$ where R is the distance between $z = 0$ and the nearest singular point which is $z = 1$. Therefore, $R = |1 - 0| = 1$.

Since $f(z)$ is analytic at $z = 3i$, it has a Taylor Series representation for all z satisfying $|z - 3i| < R$ where R is the distance between $z = 3i$ and the nearest singular point which is $z = 1$. Therefore, $R = |1 - 3i| = \sqrt{10}$.

Region of convergence about $z = 0$.



Region of convergence about $z = 3i$.



3. Find the Taylor Series of $f(z) = \dfrac{z}{1 + z^2}$ about $z = 0$ and state the region of validity. Write your answer in summation form.

Solution: The singular points of $f(z)$ are $z = i$ and $z = -i$. Since $f(z)$ is analytic at $z = 0$, it has a Taylor Series representation for all z satisfying $|z| < R$ where R is the distance between $z = 0$ and the closest singular point. Both singular points are at a distance of 1 from the origin. Therefore, the region of validity is $|z| < 1$.

We are looking for a series representation in the form

$$f(z) = \sum_{n=0}^{\infty} c_n z^n = c_0 + c_1 z + c_2 z^2 + \cdots$$

To get the Taylor Series we will write $f(z)$ as

$$f(z) = \frac{z}{1 + z^2} = z \cdot \frac{1}{1 + z^2}$$

and then use the Maclaurin Series for $\dfrac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots$ and replace z with z^2 to get

$$
\begin{aligned}
f(z) &= z \cdot \frac{1}{1+z^2} \\
f(z) &= z \cdot \left(1 - z^2 + (z^2)^2 - (z^2)^3 + \cdots\right) \\
f(z) &= z - z^3 + z^5 - z^7 + \cdots \\
f(z) &= \sum_{n=0}^{\infty} (-1)^n z^{2n+1}
\end{aligned}
$$

4. Find the Laurent Series of $f(z) = \dfrac{z}{1+z}$ about $z = 0$ in the region $1 < |z| < \infty$. Write your answer in summation form.

Solution: We are looking for a series representation in the form

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n = \cdots + \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + c_0 + c_1 z + c_2 z^2 + \cdots$$

To obtain this series we will rewrite $f(z)$ as

$$
\begin{aligned}
f(z) &= z \cdot \frac{1}{1+z} \\
f(z) &= z \cdot \frac{1}{z\left(\frac{1}{z} + 1\right)} \\
f(z) &= \frac{1}{1 + \frac{1}{z}}
\end{aligned}
$$

and then use the Maclaurin Series for $\dfrac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots$ and replace z with $\dfrac{1}{z}$ to get

$$
\begin{aligned}
f(z) &= \frac{1}{1 + \frac{1}{z}} \\
f(z) &= 1 - \frac{1}{z} + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \cdots \\
f(z) &= 1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots \\
f(z) &= \sum_{n=-\infty}^{0} (-1)^n z^n \quad \text{or} \quad \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^n}
\end{aligned}
$$

5. Determine all regions for which $f(z)$ has a Taylor Series expansion about $z = 2$. Then determine all regions for which $f(z)$ has a Laurent Series expansion about $z = 2$.

DO NOT FIND THE SERIES EXPANSIONS!

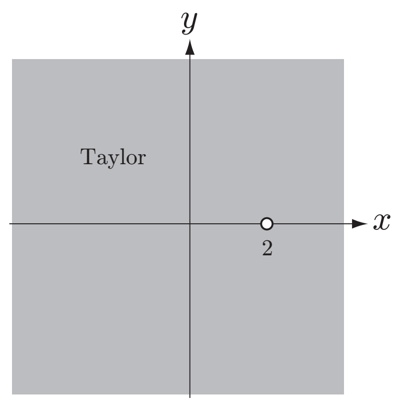
(a) $f(z) = e^z$

(b) $f(z) = \dfrac{1}{z^2 + 1}$

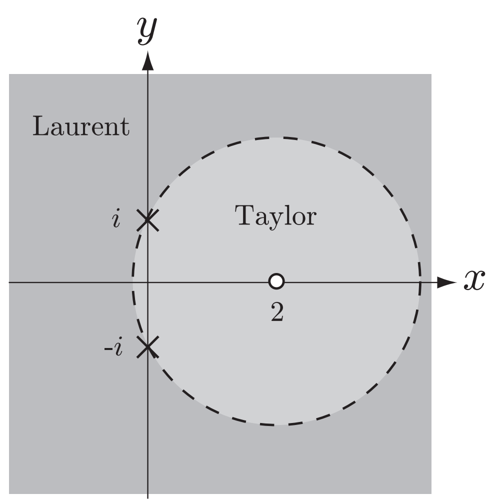
(c) $f(z) = \dfrac{1}{z(z+1)(z+2i)}$

Solution:

(a) The function is entire so it has a Taylor Series expansion that is valid for $|z-2| < \infty$.



(b) The function has singular points at $z = i$ and $z = -i$. Since $f(z)$ is analytic at $z = 2$ it has a Taylor Series expansion for all z satisfying $|z-2| < R$ where R is the distance between $z = 2$ and the nearest singular point. Both singular points are at a distance of $R = \sqrt{5}$ from $z = 2$. Therefore, $f(z)$ has a Taylor Series expansion in the region $|z-2| < \sqrt{5}$ and a Laurent Series expansion in the region $\sqrt{5} < |z-2| < \infty$.

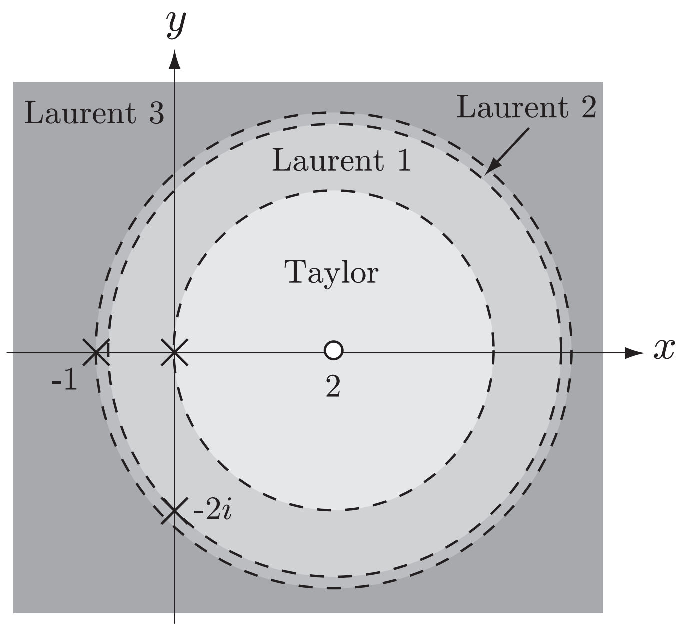


(c) The function has singular points at $z = 0$, $z = -1$, and $z = -2i$. Since $f(z)$ is analytic at $z = 2$ it has a Taylor Series expansion for all z satisfying $|z - 2| < R$ where R is the distance between $z = 2$ and the nearest singular point which is $z = 0$. The distance between these points is $R = 2$ so $f(z)$ has a Taylor Series expansion in the region $|z - 2| < 2$.

The next closest singular point is $z = -2i$. The distance between $z = 2$ and $z = -2i$ is $R = |-2i - 2| = 2\sqrt{2}$. Therefore, $f(z)$ has a Laurent Series expansion in the region $2 < |z - 2| < 2\sqrt{2}$.

The distance between $z = 2$ and the last singular point $z = -1$ is $R = |-1-2| = 3$. Therefore, $f(z)$ has another Laurent Series expansion in the region $2\sqrt{2} < |z-2| < 3$.

Finally, $f(z)$ has a third Laurent Series expansion in the region $3 < |z - 2| < \infty$.



If we were interested in finding the series expansions for $f(z) = \dfrac{1}{z(z+1))z+2i)}$ about $z = 2$, we would perform a Partial Fraction Decomposition of $f(z)$ to get

$$f(z) = \frac{1}{z(z+1)(z+2i)} = \frac{c_1}{z} + \frac{c_2}{z+1} + \frac{c_3}{z+2i}$$

where c_1, c_2, and c_3 are complex numbers. Then, on each interval we would write either a Taylor or Laurent Series for each function and it would go as follows:

$$|z-2|<2: \quad f(z) = \underbrace{\frac{c_1}{z}}_{\text{Taylor}} + \underbrace{\frac{c_2}{z+1}}_{\text{Taylor}} + \underbrace{\frac{c_3}{z+2i}}_{\text{Taylor}}$$

$$2<|z-2|<2\sqrt{2}: \quad f(z) = \underbrace{\frac{c_1}{z}}_{\text{Laurent}} + \underbrace{\frac{c_2}{z+1}}_{\text{Taylor}} + \underbrace{\frac{c_3}{z+2i}}_{\text{Taylor}}$$

$$2\sqrt{2}<|z-2|<3: \quad f(z) = \underbrace{\frac{c_1}{z}}_{\text{Laurent}} + \underbrace{\frac{c_2}{z+1}}_{\text{Taylor}} + \underbrace{\frac{c_3}{z+2i}}_{\text{Laurent}}$$

$$3<|z-2|<\infty: \quad f(z) = \underbrace{\frac{c_1}{z}}_{\text{Laurent}} + \underbrace{\frac{c_2}{z+1}}_{\text{Laurent}} + \underbrace{\frac{c_3}{z+2i}}_{\text{Laurent}}$$

6. Find the Laurent Series of $f(z) = \dfrac{1}{z^2-4}$ about $z=-1$ in the region $1<|z+1|<3$. It is not necessary to write your answer in summation form. However, you should write out sufficiently many terms so that the pattern is clear.

   Solution: First, we use the Method of Partial Fractions to rewrite the function as

$$f(z) = \frac{1}{z^2-4} = \frac{1}{4}\cdot\frac{1}{z-2} - \frac{1}{4}\cdot\frac{1}{z+2}$$

   The function $f_1(z) = \dfrac{1}{z-2}$ has a singular point at $z=2$. Since $f_1(z)$ is analytic at $z=-1$ and the distance between $z=-1$ and $z=2$ is 3, $f_1(z)$ has a Taylor Series expansion in the region $|z+1|<3$. Since we are looking for a series expansion for $f(z)$ in the annulus $1<|z+1|<3$, we will write the Taylor Series for $f_1(z)$ around $z=-1$.

$$f_1(z) = \frac{1}{z-2}$$
$$f_1(z) = \frac{1}{(z+1)-3}$$
$$f_1(z) = \frac{1}{3\left(\frac{z+1}{3} - 1\right)}$$
$$f_1(z) = -\frac{1}{3} \cdot \frac{1}{1 - \frac{z+1}{3}}$$
$$f_1(z) = -\frac{1}{3}\left(1 + \frac{z+1}{3} + \left(\frac{z+1}{3}\right)^2 + \left(\frac{z+1}{3}\right)^3 + \cdots\right)$$
$$f_1(z) = -\frac{1}{3} - \frac{z+1}{3^2} - \frac{(z+1)^2}{3^3} - \frac{(z+1)^3}{3^4} - \cdots$$

The function $f_2(z) = \dfrac{1}{z+2}$ has a singular point at $z = -2$. Since $f_2(z)$ is analytic at $z = -1$ and the distance between $z = -1$ and $z = -1$ is 1, $f_2(z)$ has a Taylor Series expansion in the region $|z+1| < 1$. However, we are interested in the series expansion of $f(z)$ in the annulus $1 < |z+1| < 3$. Therefore, we want to write the Laurent Series of $f_2(z)$ around $z = -1$.

$$f_2(z) = \frac{1}{z-2}$$
$$f_2(z) = \frac{1}{(z+1)-3}$$
$$f_2(z) = \frac{1}{(z+1)\left(1 - \frac{3}{z+1}\right)}$$
$$f_2(z) = \frac{1}{z+1} \cdot \frac{1}{1 - \frac{3}{z+1}}$$
$$f_2(z) = \frac{1}{z+1}\left(1 + \frac{3}{z+1} + \left(\frac{3}{z+1}\right)^2 + \left(\frac{3}{z+1}\right)^3 + \cdots\right)$$
$$f_2(z) = \frac{1}{z+1} + \frac{3}{(z+1)^2} + \frac{3^2}{(z+1)^3} + \frac{3^3}{(z+1)^4} + \cdots$$

Putting the series expansions for $f_1(z)$ and $f_2(z)$ back into the formula for $f(z)$ we get

$$f(z) = \frac{1}{4}f_1(z) - \frac{1}{4}f_2(z)$$

$$f(z) = \frac{1}{4}\left[-\frac{1}{3} - \frac{z+1}{3^2} - \frac{(z+1)^2}{3^3} - \cdots\right] - \frac{1}{4}\left[\frac{1}{z+1} + \frac{3}{(z+1)^2} + \frac{3^2}{(z+1)^3} + \cdots\right]$$

$$f(z) = \cdots - \frac{\frac{3^2}{4}}{(z+1)^3} - \frac{\frac{3^1}{4}}{(z+1)^2} - \frac{\frac{3^0}{4}}{(z+1)} - \frac{3^{-1}}{4} - \frac{3^{-2}}{4}(z+1) - \frac{3^{-3}}{4}(z+1)^2 - \cdots$$