MATH 417 **Homework 7**
Instructor: D. Cabrera **Due July 28**

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1. Find all singular points of the given function. For each isolated singular point, classify the point as being a removable singularity, a pole of order N (specify N), or an essential singularity.

   (a) $f(z) = \dfrac{1}{z(z-1)}$

   (b) $f(z) = \dfrac{e^z - 1}{z^3}$

   (c) $f(z) = \sin\left(\dfrac{1}{z}\right)$

   Solution:

   (a) The isolated singular points of $f(z) = \frac{1}{z(z-1)}$ are $z = 0$ and $z = 1$. The Laurent Series of $f(z)$ about $z = 0$ in the region $0 < |z| < 1$ is

   $$f(z) = -\frac{1}{z} \cdot \frac{1}{1-z}$$
   $$f(z) = -\frac{1}{z}(1 + z + z^2 + \cdots)$$
   $$f(z) = -\frac{1}{z} - 1 - \cdots$$

   Since the series begins at the $\frac{c_{-1}}{z}$ term, the singular point $z = 0$ **is a pole of order 1 or a simple pole**.

   The Laurent Series of $f(z)$ about $z = 1$ in the region $0 < |z-1| < 1$ is

   $$f(z) = \frac{1}{z-1} \cdot \frac{1}{1 + (z-1)}$$
   $$f(z) = \frac{1}{z-1}\left[1 - (z-1) + (z-1)^2 + \cdots\right]$$
   $$f(z) = \frac{1}{z-1} - 1 + \cdots$$

   Since the series begins at the $\frac{c_{-1}}{z-1}$ term, the singular point $z = 1$ **is a simple pole**.

   (b) The only isolated singular point of $f(z) = \frac{e^z-1}{z^3}$ is $z = 0$. The Laurent Series of $f(z)$ about $z = 0$ is

   $$f(z) = \frac{1}{z^3}(e^z - 1)$$
   $$f(z) = \frac{1}{z^3}\left(1 + z + \frac{z^2}{2!} + \cdots - 1\right)$$
   $$f(z) = \frac{1}{z^2} + \frac{1}{2!z} + \cdots$$

Since the series begins at the $\frac{c_{-2}}{z^2}$ term, the singular point $z = 0$ **is a pole of order 2**.

(c) The only isolated singular point of $f(z) = \sin\left(\frac{1}{z}\right)$ is $z = 0$. The Laurent Series of $f(z)$ about $z = 0$ in the region $0 < |z| < \infty$ is

$$f(z) = \sin\left(\frac{1}{z}\right)$$

$$f(z) = \frac{1}{z} - \frac{\left(\frac{1}{z}\right)^2}{3!} + \frac{\left(\frac{1}{z}\right)^5}{5!} - \cdots$$

$$f(z) = \cdots + \frac{\frac{1}{5!}}{z^5} - \frac{\frac{1}{3!}}{z^3} + \frac{1}{z}$$

There are infinitely many terms of the form $\frac{c_{-n}}{z^n}$ where n is positive. Therefore, $z = 0$ **is an essential singularity**.

2. Find all residues of $f(z) = \dfrac{1}{(z+4)(z-1)^3}$.

Solution: The singular points of $f(z)$ are $z = -4$ and $z = 1$. If we let $\phi_1(z) = \frac{1}{(z-1)^3}$, then $\phi_1(z)$ is analytic and nonzero at $z = -4$ and

$$f(z) = \frac{\phi_1(z)}{(z+4)^1}$$

Therefore, $z = -4$ is a simple pole and

$$\boxed{\operatorname*{Res}_{z=-4} f(z) = \phi_1(-4) = \left.\frac{1}{(z-1)^3}\right|_{z=-4} = -\frac{1}{125}}$$

If we let $\phi_2(z) = \frac{1}{z+4}$, then $\phi_2(z)$ is analytic and nonzero at $z = 1$ and

$$f(z) = \frac{\phi_2(z)}{(z-1)^3}$$

Therefore, $z = 1$ is a pole of order 3 and

$$\boxed{\operatorname*{Res}_{z=1} f(z) = \frac{1}{2!}\phi_2''(1) = \left.\frac{1}{2!}\frac{2}{(z+4)^3}\right|_{z=1} = \frac{1}{125}}$$

3. Evaluate $\displaystyle\int_C \frac{(z+1)^2}{z^2(z-1)}\,dz$ where C is the circle $|z| = 3$ oriented counterclockwise.

Solution: The singular points of $f(z) = \frac{(z+1)^2}{z^2(z-1)}$ are $z = 0$ and $z = 1$. Both points are interior to the circle $|z| = 3$ so the value of the integral is

$$\int_C \frac{(z+1)^2}{z^2(z-1)}\,dz = 2\pi i \left[\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=1} f(z)\right]$$

(i) If we let $\phi_1(z) = \frac{(z+1)^2}{z-1}$, then $\phi_1(z)$ is analytic and nonzero at $z = 0$ and

$$f(z) = \frac{\phi_1(z)}{z^2}$$

Therefore, $z = 0$ is a pole of order 2 and

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{1!}\phi_1'(0) = \left.\frac{z^2 - 2z - 3}{(z-1)^2}\right|_{z=0} = -3$$

(ii) If we let $\phi_2(z) = \frac{(z+1)^2}{z^2}$, then $\phi_2(z)$ is analytic and nonzero at $z = 1$ and

$$f(z) = \frac{\phi_2(z)}{(z-1)^1}$$

Therefore, $z = 1$ is a simple pole and

$$\operatorname*{Res}_{z=1} f(z) = \phi_2(1) = \frac{(1+1)^2}{1^2} = 4$$

The value of the integral is then

$$\boxed{\int_C \frac{(z+1)^2}{z^2(z-1)}\,dz = 2\pi i \left[\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=1} f(z)\right] = 2\pi i(4-3) = 2\pi i}$$

4. Evaluate $\displaystyle\int_C \frac{e^z}{\sin z}\,dz$ where C is the circle $|z - \pi| = 1$ oriented counterclockwise.

Solution: The function $f(z) = \frac{e^z}{\sin z}$ has singularities at $z = k\pi$ where $k = 0, \pm 1, \pm 2, \ldots$. The only singular point that is in the interior of the circle $|z - \pi| = 1$ is $z = \pi$. Therefore, the value of the integral is

$$\int_C \frac{e^z}{\sin z}\,dz = 2\pi i \operatorname*{Res}_{z=\pi} f(z)$$

To find the residue at $z = \pi$ we will let $p(z) = e^z$ and $q(z) = \sin z$. We recognize that both functions are analytic at $z = \pi$ and that

(1) $p(\pi) = e^\pi \neq 0$
(2) $q(\pi) = 0$
(3) $q'(\pi) = \cos\pi = -1 \neq 0$

Therefore, $z = \pi$ is a simple pole and

$$\operatorname*{Res}_{z=\pi} f(z) = \frac{p(\pi)}{q'(\pi)} = -e^{\pi}$$

The value of the integral is

$$\boxed{\int_C \frac{e^z}{\sin z}\, dz = 2\pi i \operatorname*{Res}_{z=\pi} f(z) = -2\pi i e^{\pi}}$$

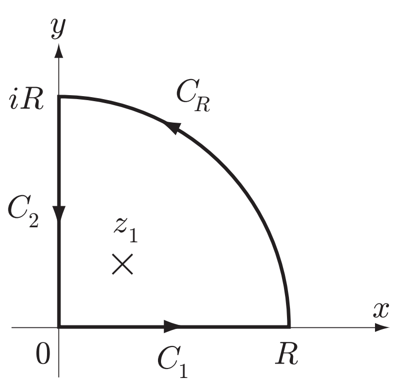
5. Show that $\displaystyle\int_0^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$.

Solution: Let's consider the contour integral

$$\int_C \frac{dz}{z^4+1}$$

where C is the contour shown below, consisting of the path along the real axis from $z = 0$ to $z = R$, the path along the quarter circle from $z = R$ to $z = iR$, and the path along the imaginary axis from $z = iR$ to $z = 0$. Then we have

$$\int_C \frac{dz}{z^4+1} = \int_{C_1} \frac{dz}{z^4+1} + \int_{C_R} \frac{dz}{z^4+1} + \int_{C_2} \frac{dz}{z^4+1}$$



(i) The integral over the simple closed contour C can be evaluated using residues. The function $f(z) = \frac{1}{z^4+1}$ has four singular points but only $z_1 = e^{\pi i/4} = \frac{\sqrt{2}}{2}(1+i)$ is interior to C so the value of the integral is

$$\int_C \frac{dz}{z^4+1} = 2\pi i \operatorname*{Res}_{z=z_1} f(z)$$

To find the residue, we'll let $p(z) = 1$ and $q(z) = z^4+1$. Notice that both functions are analytic at $z = z_1$, $p(z_1) \neq 0$, $q(z_1) = 0$, and $q'(z_1) \neq 0$. So we know that z_1 is a simple pole of $f(z)$ and that

$$\operatorname*{Res}_{z=z_1} f(z) = \frac{p(z_1)}{q'(z_1)} = \frac{1}{4z_1^3} = \frac{1}{4\sqrt{2}}(-1-i)$$

Therefore, the value of the integral over C is

$$\int_C \frac{dz}{z^4+1} = 2\pi i\left(\frac{1}{4\sqrt{2}}(-1-i)\right) = \frac{\pi}{2\sqrt{2}} - i\frac{\pi}{2\sqrt{2}}$$

(ii) The integral along C_1 is

$$\int_{C_1} \frac{dz}{z^4+1} = \int_0^R \frac{dx}{x^4+1}$$

(iii) The integral along C_2 is

$$\int_{C_2} \frac{dz}{z^4+1} = \int_R^0 \frac{i\,dy}{(iy)^4+1} = -i\int_0^R \frac{dy}{y^4+1}$$

(iv) Finally, we use the ML-Bound formula to show that the integral over C_R goes to 0. We have

$$\left|\int_{C_R} \frac{dz}{z^4+1}\right| \leq ML = \frac{1}{R^4-1}\cdot\frac{\pi R}{2} \to 0 \text{ as } R \to \infty$$

Therefore, as $R \to \infty$ we have

$$\int_C \frac{dz}{z^4+1} = \int_{C_1} \frac{dz}{z^4+1} + \int_{C_R} \frac{dz}{z^4+1} + \int_{C_2} \frac{dz}{z^4+1}$$

$$\frac{\pi}{2\sqrt{2}} - i\frac{\pi}{2\sqrt{2}} = \int_0^\infty \frac{dx}{x^4+1} + 0 - i\int_0^\infty \frac{dy}{y^4+1}$$

Taking the real part of both sides of the above equation we find that

$$\boxed{\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}}$$

Taking the imaginary part of both sides we find that

$$\int_0^\infty \frac{dy}{y^4+1} = \frac{\pi}{2\sqrt{2}}$$

which is expected since the integrals are exactly the same.

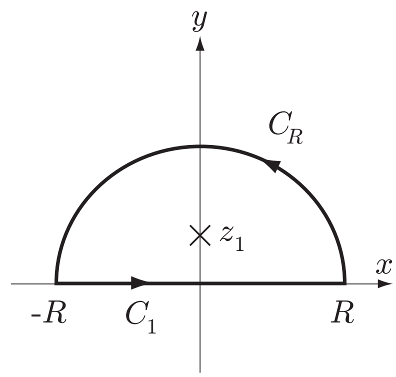
6. Show that $\displaystyle\int_{-\infty}^\infty \frac{dx}{(x^2+4)^3} = \frac{3\pi}{256}$

Solution: Let's consider the contour integral

$$\int_C \frac{dz}{(z^2+4)^3}$$

where C is the contour shown below, consisting of the path along the real axis from $z=-R$ to $z=R$ and the path along the semicircle circle from $z=R$ to $z=-R$. Then we have

$$\int_C \frac{dz}{(z^2+4)^3} = \int_{C_1} \frac{dz}{(z^2+4)^3} + \int_{C_R} \frac{dz}{(z^2+4)^3}$$



(i) The integral over the simple closed contour C can be evaluated using residues. The function $f(z) = \frac{1}{(z^2+4)^3}$ has two singular points but only $z = 2i$ is interior to C so the value of the integral is

$$\int_C \frac{dz}{(z^2+4)^3} = 2\pi i \operatorname*{Res}_{z=2i} f(z)$$

To find the residue, we'll let $\phi(z) = \frac{1}{(z+2i)^3}$. Then $\phi(z)$ is analytic and nonzero at $z = 2i$ and

$$f(z) = \frac{\phi(z)}{(z+2i)^3}$$

So we know that $z = 2i$ is a pole of order 3 and the residue there is

$$\operatorname*{Res}_{z=2i} f(z) = \frac{1}{2!}\phi''(2i) = \frac{1}{2!}\left.\frac{12}{(z+2i)^5}\right|_{z=2i} = -\frac{3i}{512}$$

Therefore, the value of the integral over C is

$$\int_C \frac{dz}{(z^2+4)^3} = 2\pi i\left(-\frac{3i}{512}\right) = \frac{3\pi}{256}$$

(ii) The integral along C_1 is

$$\int_{C_1} \frac{dz}{z^4+1} = \int_{-R}^{R} \frac{dx}{(x^2+4)^3}$$

(iii) Finally, we use the ML-Bound formula to show that the integral over C_R goes to 0. We have

$$\left|\int_{C_R} \frac{dz}{(z^2+4)^3}\right| \le \frac{1}{(R^2-4)^3}\cdot \pi R \to 0 \text{ as } R\to\infty$$

Therefore, as $R \to \infty$ we have

$$\int_C \frac{dz}{(z^2+4)^3} = \int_{C_1} \frac{dz}{(z^2+4)^3} + \int_{C_R} \frac{dz}{(z^2+4)^3}$$

$$\frac{3\pi}{256} = \lim_{R\to\infty}\int_{-R}^{R} \frac{dx}{(x^2+4)^3} + 0$$

$$\frac{3\pi}{256} = \text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^3}$$

We note that the Principal Value exists and that $f(x) = \frac{1}{(x^2+4)^3}$ is even so the Principal Value is the actual value of the integral.

$$\boxed{\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^3} = \frac{3\pi}{256}}$$