MATH 417 **Homework 8**
Instructor: D. Cabrera **Due August 4**

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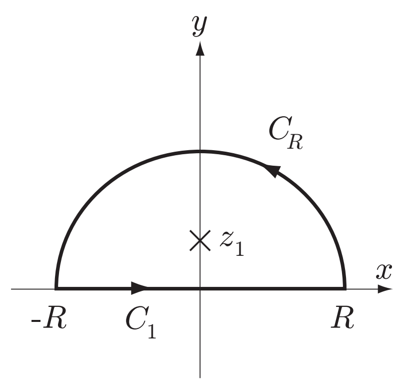
1. Compute the improper integral

$$\int_0^\infty \frac{\cos 2x}{(x^2+1)^2}\, dx$$

Solution: To evaluate the integral consider the complex integral

$$\int_C \frac{e^{i(2z)}}{(z^2+1)^2}\, dz$$

where C is the union of the contours C_1 and C_R shown below.



The complex integral can be split into two integrals:

$$\int_C \frac{e^{i(2z)}}{(z^2+1)^2}\, dz = \int_{C_1} \frac{e^{i(2z)}}{(z^2+1)^2}\, dz + \int_{C_R} \frac{e^{i(2z)}}{(z^2+1)^2}\, dz$$

Let's compute each integral in turn.

(i) The function $f(z) = \frac{e^{i(2z)}}{(z^2+1)^2}$ has singular points at i and $-i$. Only the former is inside the contour C. Therefore, the integral over C is

$$\int_C \frac{e^{i(2z)}}{(z^2+1)^2}\, dz = 2\pi i \operatorname*{Res}_{z=i} f(z)$$

To find the residue, we note that the point i is a pole of order 2. To see this, we define the function $\phi(z)$ as

$$\phi(z) = \frac{e^{i(2z)}}{(z+i)^2}$$

so that

$$f(z) = \frac{\phi(z)}{(z-i)^2}$$

Since $\phi(z)$ is analytic and nonzero at i, the point is a pole of order 2 and the residue is

$$\begin{aligned}
\operatorname*{Res}_{z=i} f(z) &= \frac{1}{1!}\phi'(i) \\
&= \left.\frac{(z+i)^2(2ie^{i(2z)}) - 2(z+i)e^{i(2z)}}{(z+i)^4}\right|_{z=i} \\
&= \frac{(i+i)^2(2ie^{2i^2}) - 2(i+i)e^{2i^2}}{(i+i)^4} \\
&= \frac{-8ie^{-2} - 4ie^{-2}}{16} \\
&= -\frac{3}{4}e^{-2}i
\end{aligned}$$

Therefore, the value of the integral over C is

$$\begin{aligned}
\int_C \frac{e^{i(2z)}}{(z^2+1)^2}\,dz &= 2\pi i \operatorname*{Res}_{z=i} f(z) \\
&= 2\pi i\left(-\frac{3}{4}e^{-2}i\right) \\
&= \frac{3\pi}{2}e^{-2}
\end{aligned}$$

(ii) The integral over C_1 is

$$\begin{aligned}
\int_{C_1} \frac{e^{i(2z)}}{(z^2+1)^2}\,dz &= \int_{-R}^{R} \frac{e^{i(2x)}}{(x^2+1)^2}\,dx \\
&= \int_{-R}^{R} \frac{\cos 2x}{(x^2+1)^2}\,dx + i\int_{-R}^{R} \frac{\sin 2x}{(x^2+1)^2}\,dx
\end{aligned}$$

(iii) Finally, we use the ML-Bound formula to evaluate the integral over C_R. First, we note that the length of the contour is $L = \pi R$. Then, we find an upper bound M on $|f(z)|$ over C_R by noting that

$$\begin{aligned}
\left|\frac{e^{i(2z)}}{(z^2+1)^2}\right| &= \frac{|e^{2i(x+iy)}|}{|z^2+1|^2} \\
&< \frac{e^{-2y}}{(R^2-1)^2} \\
&\leq \frac{1}{(R^2-1)^2} = M
\end{aligned}$$

where we used the fact that (1) $|e^{-2y}| \leq 1$ for all z on C_R since e^{-2y} takes on its maximum value on C_R when $y = 0$ and (2) $|z^2+1| \leq ||z|^2 - 1| = R^2 - 1$ using the Triangle Inequality. Thus, the modulus of the integral over C_R is bounded as follows:

$$\left|\int_{C_R} \frac{e^{i(2z)}}{(z^2+1)^2}\right| \leq \frac{\pi R}{(R^2-1)^2}$$

Putting it all together and taking the limit as $R \to \infty$ we have

$$\lim_{R\to\infty} \int_C \frac{e^{i(2z)}}{(z^2+1)^2}\, dz = \lim_{R\to\infty} \int_{C_1} \frac{e^{i(2z)}}{(z^2+1)^2}\, dz + \lim_{R\to\infty} \int_{C_R} \frac{e^{i(2z)}}{(z^2+1)^2}\, dz$$

$$\frac{3\pi}{2} e^{-2} = \lim_{R\to\infty} \int_{-R}^{R} \frac{\cos 2x}{(x^2+1)^2}\, dx + i \lim_{R\to\infty} \int_{-R}^{R} \frac{\sin 2x}{(x^2+1)^2}\, dx + 0$$

$$\frac{3\pi}{2} e^{-2} = \text{P.V.} \int_{-\infty}^{\infty} \frac{\cos 2x}{(x^2+1)^2}\, dx + i\ \text{P.V.} \int_{-R}^{R} \frac{\sin 2x}{(x^2+1)^2}\, dx$$

Taking the real parts of both sides of the above equation gives us

$$\frac{3\pi}{2} e^{-2} = \text{P.V.} \int_{-\infty}^{\infty} \frac{\cos 2x}{(x^2+1)^2}\, dx$$

Note that the integrand $f(x) = \frac{\cos 2x}{(x^2+1)^2}$ is an even function so that the principal value of the integral is the actual value. Furthermore,

$$\int_{-\infty}^{\infty} \frac{\cos 2x}{(x^2+1)^2}\, dx = 2 \int_{0}^{\infty} \frac{\cos 2x}{(x^2+1)^2}\, dx$$

So our final answer is

$$\boxed{\int_{0}^{\infty} \frac{\cos 2x}{(x^2+1)^2}\, dx = \frac{3\pi}{4} e^{-2}}$$

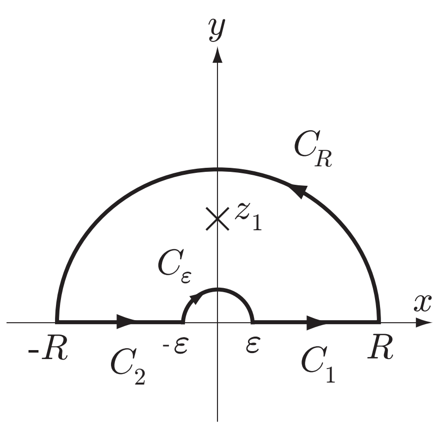
2. Show that

$$\int_0^\infty \frac{(\ln x)^2}{x^2+1}\, dx = \frac{\pi^3}{8}$$

Solution: To evaluate the integral consider the complex integral

$$\int_C \frac{(\log z)^2}{z^2+1}\, dz$$

where C is the union of the contours C_1, C_R, C_2, and C_ε shown below. Note that we take the branch cut $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ in order to avoid the contour.



The complex integral can be split into four integrals:

$$\int_C \frac{(\log z)^2}{z^2+1}\,dz = \int_{C_1} \frac{(\log z)^2}{z^2+1}\,dz + \int_{C_R} \frac{(\log z)^2}{z^2+1}\,dz + \int_{C_2} \frac{(\log z)^2}{z^2+1}\,dz + \int_{C_\varepsilon} \frac{(\log z)^2}{z^2+1}\,dz$$

Let's compute each integral in turn.

(i) The function $f(z) = \frac{(\log z)^2}{z^2+1}$ has infinitely many singular points but only $z = i$ is inside C. Therefore, the integral over C is

$$\int_C \frac{(\log z)^2}{z^2+1}\,dz = 2\pi i \operatorname*{Res}_{z=i} f(z)$$

To find the residue, we note that the point i is a simple pole. To see this, we define the function $\phi(z)$ as

$$\phi(z) = \frac{(\log z)^2}{z+i}$$

so that

$$f(z) = \frac{\phi(z)}{(z-i)^1}$$

Since $\phi(z)$ is analytic and nonzero at i, the point is a pole of order 1 and the residue is

$$\begin{aligned}\operatorname*{Res}_{z=i} f(z) &= \phi(i)\\ &= \frac{(\log i)^2}{i+i}\\ &= \frac{(\ln 1 + i\cdot\frac{\pi}{2})^2}{2i}\\ &= \frac{\pi^2}{8}i\end{aligned}$$

Therefore, the value of the integral over C is

$$\begin{aligned}\int_C \frac{(\log z)^2}{z^2+1}\,dz &= 2\pi i \operatorname*{Res}_{z=i} f(z)\\ &= 2\pi i\left(\frac{\pi^2}{8}i\right)\\ &= -\frac{\pi^3}{4}\end{aligned}$$

(ii) The integral over C_1 is parametrized by $z = re^{i(0)} = r$, $\varepsilon \le r \le R$ so that $dz = dr$ and we get

$$\begin{aligned}\int_{C_1} \frac{(\log z)^2}{z^2+1}\,dz &= \int_\varepsilon^R \frac{(\ln r + i(0))^2}{r^2+1}\,dr\\ &= \int_\varepsilon^R \frac{(\ln r)^2}{r^2+1}\,dr\end{aligned}$$

(iii) We use the ML-Bound formula to evaluate the integral over C_R. First, we note that the length of the contour is $L = \pi R$. Then, we find an upper bound M on $|f(z)|$ over C_R by noting that

$$\begin{aligned}\left|\frac{(\log z)^2}{z^2+1}\right| &= \frac{|\ln r + i\theta|^2}{|z^2+1|} \\ &\le \frac{(|\ln r| + |i\theta|)^2}{||z|^2-1|} \\ &\le \frac{(\ln R + \pi)^2}{R^2-1} = M\end{aligned}$$

where we used the Triangle Inequality on both the numerator and denominator. Thus, the modulus of the integral over C_R is bounded as follows:

$$\left|\int_{C_R} \frac{(\log z)^2}{z^2+1}\right| \le \frac{\pi R(\ln R + \pi)^2}{R^2-1}$$

We note that the right hand side of the above inequality goes to 0 as $R \to \infty$.

(iv) The integral over C_2 is parametrized by $z = re^{i\pi} = -r$, $\varepsilon \le r \le R$ so that $dz = -dr$ and we get

$$\begin{aligned}\int_{C_2} \frac{(\log z)^2}{z^2+1}\,dz &= \int_R^\varepsilon \frac{(\ln r + i\pi)^2}{r^2+1}\,(-dr) \\ &= \int_\varepsilon^R \frac{(\ln r)^2 + (2\pi \ln r)i - \pi^2}{r^2+1}\,dr \\ &= \int_\varepsilon^R \frac{(\ln r)^2}{r^2+1} - \pi^2 \int_\varepsilon^R \frac{dr}{r^2+1} + i\int_\varepsilon^R \frac{2\pi \ln r}{r^2+1}\,dr\end{aligned}$$

(v) Finally, we use the ML-Bound formula to evaluate the integral over C_ε. First, we note that the length of the contour is $L = \pi\varepsilon$. Then, we find an upper bound M on $|f(z)|$ over C_ε by noting that

$$\begin{aligned}\left|\frac{(\log z)^2}{z^2+1}\right| &= \frac{|\ln r + i\theta|^2}{|z^2+1|} \\ &\le \frac{(|\ln r| + |i\theta|)^2}{||z|^2-1|} \\ &\le \frac{(-\ln \varepsilon + \pi)^2}{1-\varepsilon^2} = M\end{aligned}$$

where we used the Triangle Inequality on both the numerator and denominator. Thus, the modulus of the integral over C_ε is bounded as follows:

$$\left|\int_{C_\varepsilon} \frac{(\log z)^2}{z^2+1}\right| \le \frac{\pi\varepsilon(-\ln \varepsilon + \pi)^2}{1-\varepsilon^2}$$

We note that the right hand side of the above inequality goes to 0 as $\varepsilon \to 0^+$.

Putting it all together and taking the limit as $\varepsilon \to 0^+$ and $R \to \infty$ we get

$$\begin{aligned}
\int_C \frac{(\log z)^2}{z^2+1}\,dz &= \int_{C_1} \frac{(\log z)^2}{z^2+1}\,dz + \int_{C_R} \frac{(\log z)^2}{z^2+1}\,dz + \int_{C_2} \frac{(\log z)^2}{z^2+1}\,dz + \int_{C_\varepsilon} \frac{(\log z)^2}{z^2+1}\,dz \\
-\frac{\pi^3}{4} &= \int_0^\infty \frac{(\ln r)^2}{r^2+1}\,dr + 0 + \int_0^\infty \frac{(\ln r)^2}{r^2+1} - \pi^2 \int_0^\infty \frac{dr}{r^2+1} + i\int_0^\infty \frac{2\pi \ln r}{r^2+1}\,dr + 0 \\
-\frac{\pi^3}{4} &= 2\int_0^\infty \frac{(\ln r)^2}{r^2+1}\,dr - \pi^2 \int_0^\infty \frac{dr}{r^2+1} + i\int_0^\infty \frac{2\pi \ln r}{r^2+1}\,dr
\end{aligned}$$

Taking the real parts of both sides we get

$$\begin{aligned}
2\int_0^\infty \frac{(\ln r)^2}{r^2+1}\,dr &= \pi^2 \int_0^\infty \frac{dr}{r^2+1} - \frac{\pi^3}{4} \\
2\int_0^\infty \frac{(\ln r)^2}{r^2+1}\,dr &= \pi^2 \left(\frac{\pi}{2}\right) - \frac{\pi^3}{4} \\
2\int_0^\infty \frac{(\ln r)^2}{r^2+1}\,dr &= \frac{\pi^3}{4} \\
\int_0^\infty \frac{(\ln r)^2}{r^2+1}\,dr &= \boxed{\frac{\pi^3}{8}}
\end{aligned}$$

Note that in the above steps we used the fact that

$$\int_0^\infty \frac{dr}{r^2+1} = \frac{\pi}{2}$$

3. Evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$$

Solution: We turn the integral into a complex integral by integrating over C, the unit circle $|z| = 1$ oriented counterclockwise, and using the substitutions

$$d\theta = \frac{dz}{iz}, \quad \sin\theta = \frac{z - \frac{1}{z}}{2i}$$

to rewrite the integral as

$$\begin{aligned}
\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} &= \int_C \frac{1}{5+4\left(\frac{z-\frac{1}{z}}{2i}\right)} \cdot \frac{dz}{iz} \\
&= \int_C \frac{1}{5iz + 2z^2 - 2}\,dz \\
&= \int_C \frac{1}{2z^2 + 5iz - 2}\,dz
\end{aligned}$$

The denominator can be factored into $(z+2i)(2z+i)$ so the singular points are $-\frac{i}{2}$ and $-2i$. Only $-\frac{i}{2}$ lies inside the contour C. Therefore, the value of the integral is

$$\int_C \frac{1}{2z^2+5iz-2}\,dz = 2\pi i \operatorname*{Res}_{z=-i/2} f(z)$$

Note that $-\frac{i}{2}$ is a simple pole of $f(z)$. Letting $p(z)=1$, $q(z)=2z^2+5iz-2$, and $q'(z)=4z+5i$, the residue is

$$\operatorname*{Res}_{z=-i/2} f(z) = \frac{p(-\frac{i}{2})}{q'(-\frac{i}{2})} = \frac{1}{4(-\frac{i}{2})+5i} = \frac{1}{3i}$$

The value of the integral is then

$$\begin{aligned}\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} &= \int_C \frac{1}{2z^2+5iz-2}\,dz \\ &= 2\pi i \operatorname*{Res}_{z=-i/2} f(z) \\ &= 2\pi i\left(\frac{1}{3i}\right) \\ &= \boxed{\frac{2\pi}{3}}\end{aligned}$$

4. Show that

$$\int_{-\pi}^{\pi} \frac{d\theta}{1+\cos^2\theta} = \pi\sqrt{2}$$

Solution: We turn the integral into a complex integral by integrating over C, the unit circle $|z|=1$ oriented counterclockwise, and using the substitutions

$$d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{z+\frac{1}{z}}{2}$$

to rewrite the integral as

$$\begin{aligned}\int_{-\pi}^{\pi} \frac{d\theta}{1+\cos^2\theta} &= \int_C \frac{1}{1+\left(\frac{z+\frac{1}{z}}{2}\right)^2}\cdot\frac{dz}{iz} \\ &= \int_C \frac{1}{1+\frac{z^2}{4}+\frac{1}{2}+\frac{1}{4z^2}}\cdot\frac{dz}{iz} \\ &= \frac{1}{i}\int_C \frac{1}{\frac{z^3}{4}+\frac{3z}{2}+\frac{1}{4z}}\,dz \\ &= \frac{4}{i}\int_C \frac{z}{z^4+6z+1}\,dz\end{aligned}$$

The singular points of the integrand are solutions to $z^4 + 6z^2 + 1 = 0$. Using the quadratic formula to solve for z^2 we have

$$z^2 = \frac{-6 \pm \sqrt{6^2 - 4(1)(1)}}{2(1)}$$
$$z^2 = \frac{-6 \pm \sqrt{32}}{2}$$
$$z^2 = -3 \pm 2\sqrt{2}$$

Taking the positive sign, we have $z_{1,2}^2 = -3+2\sqrt{2}$ which we note is negative. Therefore, two singular points are

$$z_{1,2} = \pm i\sqrt{3 - 2\sqrt{2}}$$

Taking the negative sign, we have $z_{3,4}^2 = -3 - 2\sqrt{2}$ which we note is also negative. Therefore, the other two singular points are

$$z_{3,4} = \pm i\sqrt{3 + 2\sqrt{2}}$$

Of the four singular points, only $z_{1,2}$ lie in the unit circle. These points are simple poles so we can use the formula

$$\operatorname*{Res}_{z=z_k} f(z) = \frac{p(z_k)}{q'(z_k)}$$

to find the residues at $z_{1,2}$. Letting $p(z) = z$, $q(z) = z^4 + 6z^2 + 1$, and $q'(z) = 4z^3 + 12z$ we have

$$\operatorname*{Res}_{z=z_k} f(z) = \frac{z_k}{4z_k^3 + 12z_k} = \frac{1}{4(z_k^2 + 3)}$$

To simplify the calculations here we'll note that because $z_{1,2}^2 = -3 + 2\sqrt{2}$ we have

$$z_{1,2}^2 + 3 = 2\sqrt{2}$$

Therefore, the residues at $z_{1,2}$ are

$$\operatorname*{Res}_{z=z_{1,2}} f(z) = \frac{1}{4(z_k^2 + 3)} = \frac{1}{4(2\sqrt{2})} = \frac{1}{8\sqrt{2}}$$

The value of the integral is then

$$\begin{aligned}
\int_{-\pi}^{\pi} \frac{d\theta}{1 + \cos^2\theta} &= \frac{4}{i} \int_C \frac{z}{z^4 + 6z + 1}\, dz \\
&= \frac{4}{i} \cdot 2\pi i \left(\operatorname*{Res}_{z=z_1} f(z) + \operatorname*{Res}_{z=z_2} f(z) \right) \\
&= 8\pi \left(\frac{1}{8\sqrt{2}} + \frac{1}{8\sqrt{2}} \right) \\
&= \boxed{\pi\sqrt{2}}
\end{aligned}$$

5. Use the formula for the Inverse Laplace Transform to evaluate the inverse of the function $F(s) = \dfrac{1}{(s^2+1)^2}$.

   Solution: No thanks.

6. Show that $2z^5 + 8z - 1 = 0$ has exactly four roots in the annulus $1 < |z| < 2$.

   Solution: To show that the equation has four roots in the given annulus, we will first show that it has one root inside the circle $|z| = 1$ and then show that it has five roots inside the circle $|z| = 2$.

   - Let C_1 be the circle $|z| = 1$. Define $f(z) = 8z$ and $g(z) = 2z^5 - 1$. Both functions are analytic on and inside C_1. We also have

     $$|f(z)| = |8z| = 8|z| = 8$$

     and

     $$|g(z)| = |2z^5 - 1| \leq 2|z|^5 + 1 = 3$$

     for all z on C_1. So we have established that $|f(z)| > |g(z)|$ for all z on C_1. By Rouché's Theorem, since $f(z) = 8z$ has one zero inside C_1 then so does $f(z) + g(z) = 2z^5 + 8z - 1$.

   - Now let C_2 be the circle $|z| = 2$. Define $f(z) = 2z^5$ and $g(z) = 8z - 1$. Both functions are analytic on and inside C_2. We also have

     $$|f(z)| = |2z^5| = 2|z|^5 = 2(2)^5 = 64$$

     and

     $$|g(z)| = |8z - 1| \leq 8|z| + 1 = 8(2) + 1 = 17$$

     for all z on C_2. So we have established that $|f(z)| > |g(z)|$ for all z on C_2. By Rouché's Theorem, since $f(z) = 2z^5$ has five zeros inside C_2 (counting multiplicities) then so does $f(z) + g(z) = 2z^5 + 8z - 1$.

   Finally, since $2z^5 + 8z - 1$ has one zero inside C_1 and five zeros inside C_2, it has four zeros in the annulus $1 < |z| < 2$.