Math 417 – Midterm Exam Solutions
Friday, July 11, 2008

1. Find all values of:

   (a) $\log(3-4i)$ (b) $(2+2i)^i$

   Solution:

   (a) The modulus of $z = 3-4i$ is $r = 5$ and the principal argument is $\Theta = \tan^{-1}\left(-\frac{4}{3}\right)$. Therefore, the values of $\log(3-4i)$ are

   $$\log z = \ln r + i(\Theta + 2k\pi)$$
   $$\boxed{\log(3-4i) = \ln 5 + i\left[\tan^{-1}\left(-\frac{4}{3}\right) + 2k\pi\right]}$$

   where $k = 0, \pm 1, \pm 2, \ldots$.

   (b) The values of $(2+2i)^i$ are obtained using the formula

   $$(2+2i)^i = e^{i\log(2+2i)}$$

   The modulus of $z = 2+2i$ is $r = 2\sqrt{2}$ and the principal argument is $\Theta = \frac{\pi}{4}$. Therefore,

   $$i\log(2+2i) = i\left[\ln 2\sqrt{2} + i\left(\frac{\pi}{4} + 2k\pi\right)\right]$$
   $$i\log(2+2i) = -\left(\frac{\pi}{4} + 2k\pi\right) + i\ln 2\sqrt{2}$$

   and

   $$\boxed{(2+2i)^i = e^{-(\pi/4+2k\pi)}\left[\cos\left(\ln 2\sqrt{2}\right) + i\sin\left(\ln 2\sqrt{2}\right)\right]}$$

   where $k = 0, \pm 1, \pm 2, \ldots$.

2. Complete each of the following:

   (a) Is $|e^z| = e^{|z|}$? Explain.

   (b) Explain why the following reasoning is incorrect:

   $$|e^{iz}| = |\cos z + i\sin z| = \sqrt{\cos^2 z + \sin^2 z} = 1 \quad \text{for all } z$$

   Solution:

   (a) Since $|e^z| = e^x$ and $e^{|z|} = e^{\sqrt{x^2+y^2}}$, these quantities are equal when $y = 0$ and $x \geq 0$.

(b) The reasoning fails because $|\cos z + i\sin z| \neq \sqrt{\cos^2 z + \sin^2 z}$. By definition, the modulus of a complex number $a + bi$ is $\sqrt{a^2 + b^2}$ where a and b are real numbers. In general, $\cos z$ and $\sin z$ are not real.

The modulus of e^{iz} is $|e^{iz}| = e^{-y}$ which is only equal to 1 when $y = 0$, i.e. when z is real.

3. Determine the values of z for which the function $f(z) = xe^z$ is analytic. If f is analytic at $z = 0$, then compute $f'(0)$.

Solution: Let $z = x + iy$. Then

$$f(z) = xe^{x+iy} = xe^x \cos y + ixe^x \sin y$$

We define $u(x, y) = xe^x \cos y$ and $v(x, y) = xe^x \sin y$. Their first partial derivatives are

$$u_x = (xe^x + e^x)\cos y, \quad v_y = xe^x \cos y$$
$$u_y = -xe^x \sin y, \quad v_x = (xe^x + e^x)\sin y$$

In order for the Cauchy-Riemann equations ($u_x = v_y$, $u_y = -v_x$) to be satisfied, we need

$$\begin{aligned} u_x &= v_y \\ (xe^x + e^x)\cos y &= xe^x \cos y \\ e^x \cos y &= 0 \\ \cos y &= 0 \end{aligned} \qquad\qquad \begin{aligned} u_y &= -v_x \\ -xe^x \sin y &= -(xe^x + e^x)\sin y \\ e^x \sin y &= 0 \\ \sin y &= 0 \end{aligned}$$

However, we know that $\cos y$ and $\sin y$ cannot be 0 simultaneously. Therefore, $f(z)$ is not differentiable nor analytic anywhere.

4. Consider the function $u(x, y) = e^{2x}\sin(2y) + 2x$.

(a) Show that $u(x, y)$ is harmonic in the entire z plane.

(b) Find a harmonic conjugate $v(x, y)$ of $u(x, y)$. Then express $f = u + iv$ as a function of z.

Solution:

(a) First, we can see that u has continuous derivatives of all orders for all x, y. Next, we have

$$\begin{aligned} u_x &= 2e^{2x}\sin(2y) + 2 \\ u_{xx} &= 4e^{2x}\sin(2y) \end{aligned} \qquad\qquad \begin{aligned} u_y &= 2e^{2x}\cos(2y) \\ u_{yy} &= -4e^{2x}\sin(2y) \end{aligned}$$

Clearly, we have $u_{xx} + u_{yy} = 0$ for all x, y. Therefore, $u(x, y)$ is harmonic everywhere.

(b) To find a harmonic conjugate $v(x, y)$ of $u(x, y)$ we must choose $v(x, y)$ to satisfy the Cauchy-Riemann equations:

$$\begin{aligned} v_y &= u_x \\ v_y &= 2e^{2x}\sin(2y) + 2 \end{aligned} \qquad\qquad \begin{aligned} v_x &= -u_y \\ v_x &= -2e^{2x}\cos(2y) \end{aligned}$$

The most general function that satisfies these equations is

$$\boxed{v(x, y) = -e^{2x}\cos(2y) + 2y + C}$$

5. Let C be a contour consisting of the two straight-line segments: (1) from $z = i$ to $z = 1 + i$ and (2) from $z = 1 + i$ to $z = 1 - 2i$. Compute the integral:

$$I = \int_C e^z \, dz$$

   (a) by finding a parametric representation $z(t) = x(t) + iy(t)$, $a \le t \le b$ for each line segment and computing:

$$\int_a^b f(z(t))z'(t)\, dt$$

   over each arc of the contour and

   (b) verifying the result above by using an antiderivative $F(z)$ of $f(z) = e^z$.

Solution:

   (a) See HW 4 solutions.

   (b) The function $f(z) = e^z$ is entire so it has an antiderivative $F(z) = e^z$ everywhere in the complex plane. The value of the integral is then

$$\int_C e^z \, dz = F(1 - 2i) - F(i) = e^{1-2i} = e^i$$

6. Consider the integral:

$$I = \int_C \frac{dz}{z(z+5)}$$

where C is the rectangle with corners at $z = 3 + 3i$, $z = -3 + 3i$, $z = -3 - 3i$, and $z = 3 - 3i$, oriented counterclockwise.

   (a) Find an upper bound on $|I|$. Justify your answer.

   (b) Compute the exact value of $|I|$.

Solution:

   (a) We use the ML-Bound formula to find an upper bound on $|I|$. First, the length of C is the perimeter of the square which is $L = 24$. Next, we find an upper bound on $|f(z)|$ by using some properties of moduli as follows:

$$|f(z)| = \left| \frac{1}{z(z+5)} \right| = \frac{1}{|z||z+5|}$$

   To find an upper bound on $|f(z)|$ we look for the smallest possible values of $|z|$ and $|z + 5|$ for all z on the contour. The value of $|z|$ is smallest when $z = 3, -3, 3i, -3i$ since these points are the ones on C closest to the origin. The value of $|z+5|$ is smallest when $z = -3$ since this point is the point on C closest to $z = -5$. Therefore,

$$|f(z)| = \frac{1}{|z||z+5|} \le \frac{1}{|3||-3+5|} = \frac{1}{6} = M$$

and an upper bound on $|I|$ is

$$\boxed{|I| \leq ML = \frac{1}{6} \cdot 24 = 4}$$

(b) We find the exact value of $|I|$ by first using the Method of Partial Fractions to rewrite the integral as

$$I = \int_C \frac{dz}{z(z+5)} = \frac{1}{5}\int_C \frac{dz}{z} - \frac{1}{5}\int_C \frac{dz}{z+5}$$

The function $\dfrac{1}{z+5}$ is analytic everywhere on and inside the simple closed contour C. Therefore, by the Cauchy-Goursat Theorem we have

$$\int_C \frac{dz}{z+5} = 0$$

To evaluate the first integral we notice that $\dfrac{1}{z}$ is analytic everywhere except $z = 0$, so we can deform the path into a circle of radius 1 centered at the origin. Therefore,

$$\int_C \frac{dz}{z} = 2\pi i$$

The value if I is then

$$I = \int_C \frac{dz}{z(z+5)} = \frac{1}{5}(2\pi i) - \frac{1}{5}(0) = \frac{2\pi i}{5}$$

and its modulus is

$$\boxed{|I| = \frac{2\pi}{5}}$$

which agrees with the upper bound we found in part (a).