Math 417 – Midterm Exam Solutions
Friday, July 9, 2010

Solve any 4 of Problems 1–6 and 1 of Problems 7–8. Write your solutions in the booklet provided. If you attempt more than 5 problems, you must clearly indicate which problems should be graded. Answers without justification will receive little to no credit.

1. (a) Evaluate $(-1+i)^{50}$ and write your answer in the form $a+bi$.
   (b) Find all values of $\log(-2i)$.
   (c) Find all solutions to the equation $z^3 = -8$. Write your answers in the form $a+bi$.

Solution:

(a) The modulus of $z = -1+i$ is $|z| = \sqrt{2}$ and the principal argument is $\Theta = \dfrac{3\pi}{4}$. Using DeMoivre's Theorem we have

$$(-1+i)^{50} = r^{50}\left(\cos 50\Theta + i\sin 50\Theta\right)$$
$$(-1+i)^{50} = \left(\sqrt{2}\right)^{50}\left(\cos\frac{150\pi}{4} + i\sin\frac{150\pi}{4}\right)$$
$$(-1+i)^{50} = 2^{25}\left(\cos\frac{75\pi}{2} + i\sin\frac{75\pi}{2}\right)$$

We note that the angle $\dfrac{75\pi}{2}$ is equivalent to $\dfrac{3\pi}{2}$ since $\dfrac{75\pi}{2} - 18(2\pi) = \dfrac{3\pi}{2}$. Therefore,

$$\boxed{(-1+i)^{50} = 2^{25}\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -2^{25}i}$$

(b) The modulus of $z = -2i$ is $|z| = 2$ and the principal argument is $\Theta = -\dfrac{\pi}{2}$. Using the definition of $\log z$ we have

$$\log z = \ln r + i\left(\Theta + 2k\pi\right)$$
$$\boxed{\log(-2i) = \ln 2 + i\left(-\frac{\pi}{2} + 2k\pi\right)}$$

where $k = 0, \pm1, \pm2, \ldots$.

(c) The solutions to the equation are the cube roots of -8. We use the formula:

$$z^{1/3} = r^{1/3}\left[\cos\left(\frac{\Theta + 2k\pi}{3}\right) + i\sin\left(\frac{\Theta + 2k\pi}{3}\right)\right], \quad k = 0, 1, 2$$

The modulus of $z = -8$ is $|z| = r = 8$ and the principal argument is $\Theta = \pi$. Therefore, the solutions are

$$z^{1/3} = 8^{1/3}\left[\cos\left(\frac{\pi + 2(0)\pi}{3}\right) + i\sin\left(\frac{\pi + 2(0)\pi}{3}\right)\right] = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \boxed{1 + i\sqrt{3}}$$

$$z^{1/3} = 8^{1/3}\left[\cos\left(\frac{\pi + 2(1)\pi}{3}\right) + i\sin\left(\frac{\pi + 2(1)\pi}{3}\right)\right] = 2\left(\cos\pi + i\sin\pi\right) = \boxed{-2}$$

$$z^{1/3} = 8^{1/3}\left[\cos\left(\frac{\pi + 2(2)\pi}{3}\right) + i\sin\left(\frac{\pi + 2(2)\pi}{3}\right)\right] = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right) = \boxed{1 - i\sqrt{3}}$$

2. (a) Sketch the set of points defined by the inequality $|z + 2| \leq |z|$.

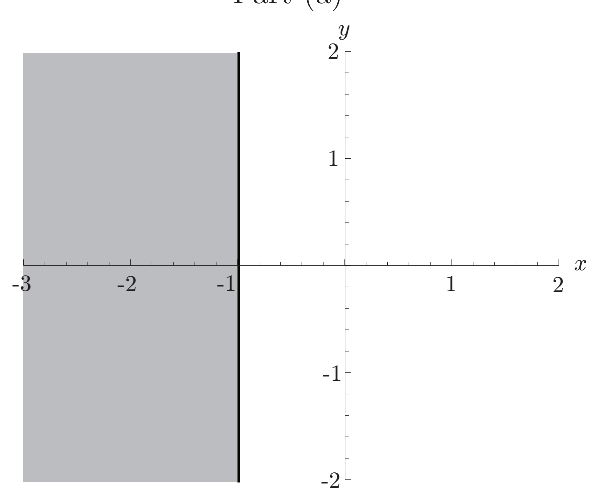
   (b) Sketch the image of the set of points in the z-plane defined by $-\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$, $0 \leq y < \infty$ under the transformation $w = \sin z$.

Solution:

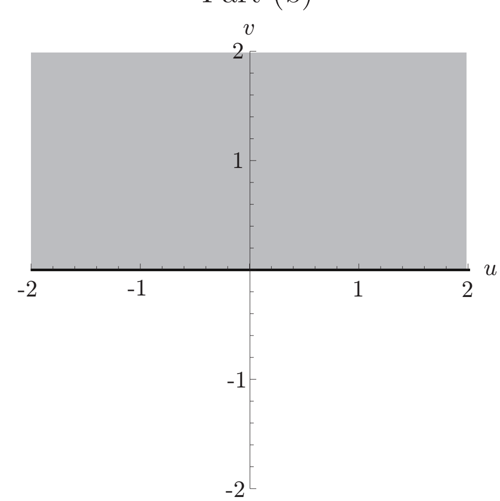
(a) Letting $z = x + iy$ we have

$$\begin{aligned}
|z + 2| &\leq |z| \\
|x + iy + 2| &\leq |x + iy| \\
|(x + 2) + iy| &\leq |x + iy| \\
\sqrt{(x + 2)^2 + y^2} &\leq \sqrt{x^2 + y^2} \\
(x + 2)^2 + y^2 &\leq x^2 + y^2 \\
x^2 + 4x + 4 + y^2 &\leq x^2 + y^2 \\
4x + 4 &\leq 0 \\
x &\leq -1
\end{aligned}$$

Part (a)



Part (b)



(b) First, we want to write $\sin z$ in terms of x and y.

$$w = \sin(z) = \sin x \cosh y + i \cos x \sinh y$$

We consider w to be complex and write $w = u_i v$ so that

$$\begin{aligned} u &= \sin x \cosh y \\ v &= \cos x \sinh y \end{aligned}$$

Now let's consider the transformation of each boundary.

i. For $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, $y = 0$ we have

$$\begin{aligned} u &= \sin x \cosh 0 = \sin x \\ v &= \cos x \sinh 0 = 0 \end{aligned}$$

Thus, the transformation is the line segment $-1 \leq u \leq 1$, $v = 0$.

ii. For $x = -\frac{\pi}{2}$, $0 \leq y < \infty$ we have

$$\begin{aligned} u &= \sin\left(-\frac{\pi}{2}\right) \cosh y = -\cosh y \\ v &= \cos\left(-\frac{\pi}{2}\right) \sinh y = 0 \end{aligned}$$

Thus, the transformation is the line segment $-\infty < u \leq -1$, $v = 0$.

iii. For $x = \frac{\pi}{2}$, $0 \leq y < \infty$ we have

$$\begin{aligned} u &= \sin \frac{\pi}{2} \cosh y = \cosh y \\ v &= \cos \frac{\pi}{2} \sinh y = 0 \end{aligned}$$

Thus, the transformation is the line segment $1 \leq u < \infty$, $v = 0$.

Putting these together, the boundary of the given region is transformed into the entire u-axis in the w-plane. Now take a test point in the given region, say, $z = 0 + i$. Then we have

$$\begin{aligned} u &= \sin 0 \cosh 1 = 0 \\ v &= \cos 0 \sinh 1 = \frac{e - e^{-1}}{2} > 0 \end{aligned}$$

The transformation of $z = 0+i$ is $w = 0 + i\left(\frac{e - e^{-1}}{2}\right)$ which is in the upper half of the w-plane.

3. In each part below, z is a complex number.

   (a) Show that $|z^2| = |z|^2$ for all z.

   (b) Find the values of z, if any, for which $\overline{e^z} = e^{\overline{z}}$.

Solution:

(a) This can be proven in one of two ways. If we let $z = x + iy$ then

$$\begin{aligned}
|z^2| &= |(x+iy)^2| \\
|z^2| &= |(x^2-y^2)+i(2xy)| \\
|z^2| &= \sqrt{(x^2-y^2)^2+(2xy)^2} \\
|z^2| &= \sqrt{x^4-2x^2y^2+y^4+4x^2+y^2} \\
|z^2| &= \sqrt{x^4+2x^2y^2+y^4} \\
|z^2| &= \sqrt{(x^2+y^2)^2} \\
|z^2| &= x^2+y^2 \\
|z^2| &= |z|^2
\end{aligned}$$

for all z. If, instead, we let $z = re^{i\theta}$ then

$$|z^2| = |r^2e^{2i\theta}| = r^2|e^{2i\theta}| = r^2 = |z|^2$$

for all z.

(b) If we let $z = x + iy$ then

$$\begin{aligned}
\overline{e^z} &= \overline{e^{x+iy}} \\
\overline{e^z} &= \overline{e^x e^{iy}} \\
\overline{e^z} &= \overline{e^x(\cos y + i\sin y)} \\
\overline{e^z} &= e^x(\cos y - i\sin y) \\
\overline{e^z} &= e^x e^{-iy} \\
\overline{e^z} &= e^{x-iy} \\
\overline{e^z} &= e^{\overline{z}}
\end{aligned}$$

for all z.

4. Consider the function $f(z) = z^2\overline{z}$.

   (a) Write the function in the form $f(z) = u(x,y) + iv(x,y)$.

   (b) Find all values of z for which $f'(z)$ exists.

Solution:

(a) Let $z = x + iy$. Then $z = x - iy$ and we have

$$\begin{aligned}
f(z) &= z^2\overline{z} \\
f(z) &= (x+iy)^2(x-iy) \\
f(z) &= \left[(x^2-y^2)+i(2xy)\right](x-iy) \\
f(z) &= x(x^2-y^2)+2xy^2+i\left[-y(x^2-y^2)+2x^2y\right] \\
f(z) &= x^3+xy^2+i(y^3+x^2y)
\end{aligned}$$

(b) Let $u(x,y) = x^3 + xy^2$ and $v(x,y) = y^3 + x^2y$. Both $u(x,y)$ and $v(x,y)$ have continuous derivatives of all orders everywhere in the complex plane. The first partial derivatives are

$$\begin{aligned}
u_x &= 3x^2+y^2, \quad v_y = 3y^2+x^2 \\
u_y &= 2xy, \quad v_x = 2xy
\end{aligned}$$

In order for the Cauchy-Riemann equations to be satisfied we need

$$\begin{aligned}
u_x &= v_y & \qquad u_y &= -v_x \\
3x^2+y^2 &= 3y^2+x^2 & \qquad 2xy &= -2xy \\
2x^2 &= 2y^2 & \qquad 4xy &= 0 \\
x &= \pm y & \qquad xy &= 0
\end{aligned}$$

The second equation says that either $x = 0$ or $y = 0$. If $x = 0$ then the first equation says that $= 0$. If $y = 0$ then the first equation says that $x = 0$. Thus, the C-R equations are only satisfied when $z = 0$ and $f'(z)$ exists only when $z = 0$.

5. Determine the values of z for which the function $f(z) = \bar{z}e^x$ is differentiable and evaluate $f'(z)$ at each point. At what points, if any, is $f(z)$ analytic?

Solution: Let $z = x + iy$. Then

$$\begin{aligned}
f(z) &= \bar{z}e^x \\
f(z) &= (x-iy)e^x \\
f(z) &= xe^x + i(-ye^x)
\end{aligned}$$

We have $u(x,y) = xe^x$ and $v(x,y) = -ye^x$. These functions have continous derivatives of all orders everywhere in the complex plane. The first partial derivatives are

$$\begin{aligned}
u_x &= xe^x + e^x, \quad v_y = -e^x \\
u_y &= 0, \quad v_x = -ye^x
\end{aligned}$$

In order for the Cauchy-Riemann equations to be satisfied we need

$$\begin{aligned} u_x &= v_y \\ xe^x + e^x &= -e^x \\ xe^x + 2e^x &= 0 \\ e^x(x+2) &= 0 \end{aligned} \qquad\qquad \begin{aligned} u_y &= -v_x \\ 0 &= -ye^x \\ ye^x &= 0 \end{aligned}$$

Since $e^x > 0$ for all x, the first equation tells us that $x = -2$ and the second equation tells us that $y = 0$. Therefore, the C-R equations are only satisfied when $z = -2$ and $f'(z)$ exists only when $z = -2$. There is no neighborhood of $z = -2$ throughout which $f'(z)$ exists. Thus, $f(z)$ is analytic nowhere.

6. Consider the function $u(x, y) = xy^3 - x^3y + 2x - 6y$.

   (a) Show that $u(x, y)$ is harmonic in the entire complex plane.

   (b) Find a harmonic conjugate $v(x, y)$ of $u(x, y)$.

Solution:

(a) The function $u(x, y)$ has continuous derivatives of all orders everywhere in the complex plane. The first and second partial derivatives are

$$\begin{aligned} u_x &= y^3 - 3x^2y + 2, \quad u_{xx} = -6xy \\ u_y &= 3xy^2 - x^3 - 6, \quad u_{yy} = 6xy \end{aligned}$$

We can see that $u_{xx} + u_{yy} = -6xy + 6xy = 0$ for all x, y. Therefore, $u(x, y)$ is harmonic in the entire complex plane.

(b) A harmonic conjugate $v(x, y)$ of $u(x, y)$ must satisfy the Cauchy-Riemann equations.

$$\begin{aligned} v_y &= u_x \\ v_y &= y^3 - 3x^2y + 2 \end{aligned} \qquad\qquad \begin{aligned} v_x &= -u_y \\ v_x &= -3xy^2 + x^3 + 6 \end{aligned}$$

Integrating the first equation with respect to y we have

$$\begin{aligned} \int v_y \, dy &= \int (y^3 - 3x^2y + 2) \, dy \\ v(x, y) &= \frac{1}{4}y^4 - \frac{3}{2}x^2y^2 + 2y + \phi(x) \end{aligned}$$

Differentiating this equation with respect to x and setting the result equation to the equation for v_x above we get

$$\begin{aligned}
\frac{\partial}{\partial x} v(x,y) &= v_x \\
\frac{\partial}{\partial x}\left(\frac{1}{4}y^4 - \frac{3}{2}x^2y^2 + 2y + \phi(x)\right) &= -3xy^2 + x^3 + 6 \\
-3xy^2 + \phi'(x) &= -3xy^2 + x^3 + 6 \\
\phi'(x) &= x^3 + 6 \\
\phi(x) &= \int (x^3 + 6)\,dx \\
\phi(x) &= \frac{1}{4}x^4 + 6x + C
\end{aligned}$$

Therefore, the family of harmonic conjugates of $u(x, y)$ are

$$\boxed{v(x,y) = \frac{1}{4}(x^4 + y^4) - \frac{3}{2}x^2y^2 + 2y + 6x + C}$$

7. Consider the integral

$$I = \int_C \left(\bar{z}^2 - \bar{z}\right)\,dz$$

where C is the circle $|z| = 2$ oriented counterclockwise.

(a) Use the ML-Bound formula to find an upper bound on $|I|$.

(b) Find the exact value of $|I|$.

Solution:

(a) First, the length of the contour is $L = 2\pi r = 2\pi(2) = 4\pi$. Next, we find an upper bound on $|f(z)|$ for all z on C using the Triangle Inequality.

$$\left|\bar{z}^2 - \bar{z}\right| \le |\bar{z}^2| + |\bar{z}| = |z|^2 + |z| = 2^2 + 2 = 6$$

Therefore, we let $M = 6$ and we get the following upper bound on $|I|$:

$$|I| \le ML = 6(4\pi) = 24\pi$$

(b) The function $f(z) = \bar{z}^2 - \bar{z}$ is analytic nowhere. So we have to parametrize the contour. Let $z(t) = 2e^{it}$ where $0 \le t \le 2\pi$. Then $\bar{z} = 2e^{-it}$, $z'(t) = 2ie^{it}$, and

$$\begin{aligned}
\int_C f(z)\,dz &= \int_a^b f(z(t))z'(t)\,dt \\
\int_C (\bar{z}^2 - \bar{z})\,dz &= \int_0^{2\pi} \left(4e^{-2it} - 2e^{-it}\right)\left(2ie^{it}\right) dt \\
&= 4i \int_0^{2\pi} \left(2e^{-it} - 1\right)\, dt \\
&= 4i \left[-\frac{2}{i} e^{-it} - t \right]_0^{2\pi} \\
&= 4i \left[-\frac{2}{i} e^{-2\pi i} - 2\pi + \frac{2}{i} e^0 + 0 \right] \\
&= 4i \left[-\frac{2}{i} - 2\pi + \frac{2}{i} \right] \\
&= -8\pi i
\end{aligned}$$

So the exact value of $|I|$ is $|I| = 8\pi$.

8. Evaluate the integral

$$\int_C |z|^2\,dz$$

where the contour C is

(a) the line segment with initial point -1 and final point i

(b) the arc of the unit circle $|z| = 1$ traversed in the clockwise direction with initial point -1 and final point i.

Why don't the two results agree?

Solution:

(a) A parametrization of C is $z(t) = t + i(t+1)$ where $-1 \le t \le 0$. Then $z'(t) = 1 + i$

and we have

$$\begin{aligned}
\int_C f(z)\,dz &= \int_a^b f(z(t))z'(t)\,dt \\
\int_C |z|^2\,dz &= \int_{-1}^0 \left[t^2 + (t+1)^2\right](1+i)\,dt \\
&= (1+i)\int_{-1}^0 (2t^2 + 2t + 1)\,dt \\
&= (1+i)\left[\frac{2}{3}t^3 + t^2 + t\right]_{-1}^0 \\
&= (1+i)\left[0 - \left(-\frac{2}{3} + 1 - 1\right)\right] \\
&= \frac{2}{3}(1+i)
\end{aligned}$$

(b) A parametrization of C is $z(t) = e^{it}$ where $\frac{\pi}{2} \leq t \leq \pi$. Then $z'(t) = ie^{it}$ and

$$\begin{aligned}
\int_C f(z)\,dz &= \int_a^b f(z(t))z'(t)\,dt \\
\int_C |z|^2\,dz &= \int_\pi^{\pi/2} |e^{it}|^2(ie^{it})\,dt \\
&= \int_\pi^{\pi/2} ie^{it}\,dt \\
&= e^{it}\Big|_\pi^{\pi/2} \\
&= e^{i\pi/2} - e^{i\pi} \\
&= i + 1
\end{aligned}$$

The results do not agree because $f(z)$ is analytic nowhere so the integral is not path independent.