

Math 417 – Sections 14–17 Solutions

2. (b) Show that $\lim_{z \rightarrow z_0} (z^2 + c) = z_0^2 + c$.

Given $\varepsilon > 0$, we must find a $\delta > 0$ such that:

$$|(z^2 + c) - (z_0^2 + c)| < \varepsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta$$

We work with the first of the above inequalities:

$$\begin{aligned} |(z^2 + c) - (z_0^2 + c)| &= |z^2 - z_0^2| \\ &= |(z - z_0)(z + z_0)| \\ &= |z - z_0||z + z_0| \\ &= |z - z_0||z - z_0 + 2z_0| \\ &\leq |z - z_0|(|z - z_0| + 2|z_0|) \end{aligned}$$

Let us now choose δ such that the following inequality is satisfied:

$$|z - z_0|(|z - z_0| + 2|z_0|) < \varepsilon \tag{1}$$

Let $x = |z - z_0|$ and solve the equation $x(x + 2|z_0|) = \varepsilon$:

$$\begin{aligned} x(x + 2|z_0|) &= \varepsilon \\ x^2 + 2|z_0|x - \varepsilon &= 0 \\ x &= \frac{-2|z_0| + \sqrt{4|z_0|^2 + 4\varepsilon}}{2} \\ x &= -|z_0| + \sqrt{|z_0|^2 + \varepsilon} \end{aligned}$$

Therefore, the solution to the inequality (1) is:

$$|z - z_0| < -|z_0| + \sqrt{|z_0|^2 + \varepsilon}$$

Hence, we choose δ as follows:

$$\delta = -|z_0| + \sqrt{|z_0|^2 + \varepsilon}$$

Since this ensures that inequality (1) is satisfied, it follows that:

$$|(z^2 + c) - (z_0^2 + c)| \leq |z - z_0|(|z - z_0| + 2|z_0|) < \varepsilon$$

5. Let $z = x + iy$. Then the function $f(z)$ is:

$$f(z) = \frac{z^2}{\bar{z}^2} = \frac{(x + iy)^2}{(x - iy)^2}$$

We compute the limit of the function as $z \rightarrow 0$ along two different paths:

$$\text{Path 1 : } y = 0$$

$$\text{Path 2 : } y = x$$

Along Path 1 we have:

$$\lim_{z \rightarrow 0} f(z) = \lim_{x \rightarrow 0^+} \frac{(x + i0)^2}{(x - i0)^2} = \lim_{x \rightarrow 0^+} \frac{x^2}{x^2} = 1$$

Along Path 2 we have:

$$\lim_{z \rightarrow 0} f(z) = \lim_{x \rightarrow 0^+} \frac{(x + ix)^2}{(x - ix)^2} = \lim_{x \rightarrow 0^+} \frac{x^2(1 + i)^2}{x^2(1 - i)^2} = \lim_{x \rightarrow 0^+} \frac{2i}{-2i} = -1$$

Therefore, since the value of the limit is different along each path, the limit does not exist.

10. (a) To show that:

$$\lim_{z \rightarrow \infty} \frac{4z^2}{(z - 1)^2} = 4$$

we must show that:

$$\lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) = 4$$

i.e. statement (2) on p. 49. We show this as follows:

$$\begin{aligned} \lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) &= \lim_{z \rightarrow 0} \frac{4\left(\frac{1}{z}\right)^2}{\left(\frac{1}{z} - 1\right)^2} \\ &= \lim_{z \rightarrow 0} \frac{\frac{4}{z^2}}{\frac{1}{z^2} - \frac{2}{z} + 1} \\ &= \lim_{z \rightarrow 0} \frac{4}{1 - 2z + z^2} \\ &= 4 \end{aligned}$$

Therefore, by statement (2), we have:

$$\lim_{z \rightarrow \infty} \frac{4z^2}{(z - 1)^2} = 4$$

(b) To show that:

$$\lim_{z \rightarrow 1} \frac{1}{(z - 1)^3} = \infty$$

we must show that:

$$\lim_{z \rightarrow 1} \frac{1}{f(z)} = 0$$

i.e. statement (1) on p. 49. We show this as follows:

$$\begin{aligned} \lim_{z \rightarrow 1} \frac{1}{f(z)} &= \lim_{z \rightarrow 1} (z - 1)^3 \\ &= 0 \end{aligned}$$

Therefore, by statement (1), we have:

$$\lim_{z \rightarrow 1} \frac{1}{(z - 1)^3} = \infty$$