

Math 417 – Sections 20–22 Solutions

1. (b) $f(z) = z - \bar{z}$

Let $z = x + iy$. Then $f(z) = (x + iy) - (x - iy) = 0 + i(2y) \Rightarrow u(x, y) = 0$ and $v(x, y) = 2y$. The first partials are:

$$\begin{aligned} u_x &= 0, & u_y &= 0 \\ v_x &= 0, & v_y &= 2 \end{aligned}$$

The first of the Cauchy-Riemann equations ($u_x = v_y$) is never satisfied. Therefore, $f'(z)$ does not exist anywhere.

- (c) $f(z) = 2x + ixy^2$

We have $u(x, y) = 2x$ and $v(x, y) = xy^2$. The first partials are:

$$\begin{aligned} u_x &= 2, & u_y &= 0 \\ v_x &= y^2, & v_y &= 2xy \end{aligned}$$

The first of the Cauchy-Riemann equations ($u_x = v_y$) is satisfied when $xy = 1$. However, the second of the Cauchy-Riemann equations ($u_y = -v_x$) is satisfied only when $y = 0$. There is no x that satisfies $xy = 1$ when $y = 0$. Therefore, $f'(z)$ does not exist anywhere.

2. (b) $f(z) = e^{-x}e^{-iy}$.

First, we have:

$$f(z) = e^{-x}e^{-iy} = e^{-x}(\cos(-y) + i\sin(-y)) = e^{-x}\cos y + i(-e^{-x}\sin y)$$

Therefore, $u(x, y) = e^{-x}\cos y$ and $v(x, y) = -e^{-x}\sin y$. The first partials are:

$$\begin{aligned} u_x &= -e^{-x}\cos y, & u_y &= -e^{-x}\sin y \\ v_x &= e^{-x}\sin y, & v_y &= -e^{-x}\cos y \end{aligned}$$

The Cauchy-Riemann equations ($u_x = v_y$, $u_y = -v_x$) are satisfied for all x, y . Furthermore, the first partials exist everywhere and are continuous everywhere. Therefore, $f'(z)$ exists everywhere and:

$$f'(z) = (-e^{-x}\cos y) + i(e^{-x}\sin y)$$

Now consider the function $g(z) = f'(z)$. Let $u(x, y) = -e^{-x}\cos y$ and $v(x, y) = e^{-x}\sin y$. The first partials are:

$$\begin{aligned} u_x &= e^{-x}\cos y, & u_y &= e^{-x}\sin y \\ v_x &= -e^{-x}\sin y, & v_y &= e^{-x}\cos y \end{aligned}$$

The Cauchy-Riemann equations ($u_x = v_y$, $u_y = -v_x$) are satisfied for all x, y . Furthermore, the first partials exist everywhere and are continuous everywhere. Therefore, $g'(z) = f''(z)$ exists everywhere and:

$$g'(z) = f''(z) = (e^{-x}\cos y) + i(-e^{-x}\sin y)$$

3. (c) $f(z) = z \operatorname{Im} z$

Let $z = x + iy$. Then $f(z) = (x + iy)(y) = xy + iy^2 \Rightarrow u(x, y) = xy$ and $v(x, y) = y^2$. The first partials are:

$$\begin{aligned} u_x &= y, & u_y &= x \\ v_x &= 0, & v_y &= 2y \end{aligned}$$

The Cauchy-Riemann equations are satisfied only when $x = 0$ and $y = 0$. Furthermore, the first partials exist everywhere and are continuous at $(0, 0)$. Therefore, the derivative of $f(z)$ at $z = 0$ is:

$$\boxed{f'(0) = 0}$$

4. (c) For the function $f(z) = e^{-\theta} \cos(\ln r) + ie^{-\theta} \sin(\ln r)$ ($r > 0$, $0 < \theta < 2\pi$), we have:

$$\begin{aligned} u(r, \theta) &= e^{-\theta} \cos(\ln r) \\ v(r, \theta) &= e^{-\theta} \sin(\ln r) \end{aligned}$$

The first derivatives are:

$$\begin{aligned} u_r &= -\frac{1}{r} e^{-\theta} \sin(\ln r), & u_\theta &= -e^{-\theta} \cos(\ln r) \\ v_r &= \frac{1}{r} e^{-\theta} \cos(\ln r), & v_\theta &= -e^{-\theta} \sin(\ln r) \end{aligned}$$

These satisfy the Cauchy-Riemann equations for all r and θ in the domain of definition:

$$ru_r = v_\theta, \quad u_\theta = -rv_r$$

Furthermore, the first partials exist everywhere and are continuous everywhere in the domain of definition. Therefore, $f(z)$ is differentiable in the given domain and:

$$\begin{aligned} f'(z) &= e^{-i\theta} (u_r + iv_r) \\ &= e^{-i\theta} \left(-\frac{1}{r} e^{-\theta} \sin(\ln r) + \frac{i}{r} e^{-\theta} \cos(\ln r) \right) \\ &= \frac{i}{r} e^{-i\theta} (ie^{-\theta} \sin(\ln r) + e^{-\theta} \cos(\ln r)) \\ &= \frac{i}{re^{i\theta}} (e^{-\theta} \cos(\ln r) + ie^{-\theta} \sin(\ln r)) \\ &= \boxed{\frac{i}{z} f(z)} \end{aligned}$$

5. $f(z) = x^3 + i(1 - y)^3$

Let $u(x, y) = x^3$ and $v(x, y) = (1 - y)^3$. The first partials are:

$$\begin{aligned} u_x &= 3x^2, & u_y &= 0 \\ v_x &= 0, & v_y &= -(1 - y)^2 \end{aligned}$$

The second of the Cauchy-Riemann equations ($u_y = -v_x$) is always satisfied. In order for the first of the Cauchy-Riemann equations ($u_x = v_y$) to be satisfied, we must have:

$$3x^2 = -(1 - y)^2$$

This can only happen when both sides of the equation are 0 $\Rightarrow x = 0$ and $y = 1$. Furthermore, the first partials exist everywhere and are continuous everywhere. Therefore, it is legitimate to write:

$$f'(z) = u_x + iv_x = 3x^2$$

only when $z = 0 + i1 = i$.