

# Math 417 – Sections 23 Solutions

1. (d) Letting  $z = x + iy$  in the function  $f(z) = (z^2 - 2)e^{-x}e^{-iy}$  we have:

$$\begin{aligned} f(z) &= [(x^2 - y^2) + i(2xy)] e^{-x}(\cos y - i \sin y) \\ &= e^{-x}[(x^2 - y^2) \cos y + 2xy \sin y + i(2xy \cos y - (x^2 - y^2) \sin y)] \end{aligned}$$

Then we have:

$$\begin{aligned} u(x, y) &= e^{-x}[(x^2 - y^2) \cos y + 2xy \sin y] \\ v(x, y) &= e^{-x}[2xy \cos y - (x^2 - y^2) \sin y] \end{aligned}$$

The first partials are:

$$\begin{aligned} u_x &= -e^{-x}[(x^2 - y^2) \cos y + 2xy \sin y] + e^{-x}[2x \cos y + 2y \sin y] \\ v_y &= e^{-x}[2x \cos y - 2xy \sin y + 2y \sin y - (x^2 - y^2) \cos y] \\ &= -e^{-x}[(x^2 - y^2) \cos y + 2xy \sin y] + e^{-x}[2x \cos y + 2y \sin y] \\ u_y &= e^{-x}[-2y \cos y - (x^2 - y^2) \cos y + 2x \sin y + 2xy \cos y] \\ &= e^{-x}[2xy \cos y - (x^2 - y^2) \cos y] - e^{-x}[2y \cos y - 2x \sin y] \\ v_x &= -e^{-x}[2xy \cos y - (x^2 - y^2) \sin y] + e^{-x}[2y \cos y - 2x \sin y] \end{aligned}$$

It can be seen that the Cauchy-Riemann equations ( $u_x = v_y$ ,  $u_y = -v_x$ ) are satisfied. Furthermore, the first partials exist everywhere and are continuous everywhere. Therefore,  $f(z)$  is entire.

2. (a) From  $f(z) = xy + iy$ , we have:

$$\begin{aligned} u(x, y) &= xy \\ v(x, y) &= y \end{aligned}$$

The first partials are:

$$\begin{aligned} u_x &= y, & v_y &= 1 \\ u_y &= x, & v_x &= 0 \end{aligned}$$

The Cauchy-Riemann equations are satisfied only when  $y = 1$  and  $x = 0$ . Therefore,  $f'(z)$  exists only at the point  $z = 0 + i1 = i$ . Since  $f(z)$  is not analytic in any neighborhood of  $z = i$ , it is analytic nowhere.

4. (b) The singular points of  $f(z) = \frac{z^3 + i}{z^2 - 3z + 2}$  occur when the denominator is 0:

$$\begin{aligned} z^2 - 3z + 2 &= 0 \\ \Rightarrow z &= 2, z = 1 \end{aligned}$$

$f'(z)$  exists everywhere except at the above points since  $f(z)$  is a rational function.