

Math 417 – Section 25 Solutions

1. (b) For the function $u(x, y) = 2x - x^3 + 3xy^2$ we have:

$$\begin{aligned} u_x &= 2 - 3x^2 + 3y^2 & u_{xx} &= -6x \\ u_y &= 6xy & u_{yy} &= 6x \end{aligned}$$

Since $u_{xx} + u_{yy} = -6x + 6x = 0$ and $u(x, y)$ has continuous first and second partial derivatives everywhere, $u(x, y)$ is harmonic.

To find a harmonic conjugate of $u(x, y)$, we use the Cauchy-Riemann equations:

$$\begin{aligned} v_y &= u_x = 2 - 3x^2 + 3y^2 \\ v_x &= -u_y = -6xy \end{aligned}$$

Integrating the first of the above equations with respect to y we get:

$$v(x, y) = 2y - 3x^2y + y^3 + f(x)$$

Differentiating the above equation with respect to x and setting the result equal to the second of the Cauchy-Riemann equations, we get:

$$\begin{aligned} v_x &= -6xy + f'(x) = -6xy \\ f'(x) &= 0 \\ f(x) &= C \end{aligned}$$

Therefore, the harmonic conjugates of $u(x, y)$ are:

$$v(x, y) = 2y - 3x^2y + y^3 + C$$

- (d) For the function $u(x, y) = \frac{y}{x^2 + y^2}$ we have:

$$\begin{aligned} u_x &= -\frac{2xy}{(x^2 + y^2)^2} & u_{xx} &= -\left[\frac{(x^2 + y^2)^2(2y) - 4x(2xy)(x^2 + y^2)}{(x^2 + y^2)^4} \right] = \frac{6x^2y - 2y^3}{(x^2 + y^2)^3} \\ u_y &= \frac{x^2 - y^2}{(x^2 + y^2)^2} & u_{yy} &= \frac{(x^2 + y^2)^2(-2y) - 4y(x^2 - y^2)(x^2 + y^2)}{(x^2 + y^2)^4} = \frac{-6x^2y + 2y^3}{(x^2 + y^2)^3} \end{aligned}$$

Since $u_{xx} + u_{yy} = 0$ and $u(x, y)$ has continuous first and second partial derivatives everywhere in its domain (which does not include $z = 0$), $u(x, y)$ is harmonic.

To find a harmonic conjugate of $u(x, y)$, we use the Cauchy-Riemann equations:

$$\begin{aligned} v_y &= u_x = -\frac{2xy}{(x^2 + y^2)^2} \\ v_x &= -u_y = -\frac{x^2 - y^2}{(x^2 + y^2)^2} \end{aligned}$$

Integrating the first of the above equations with respect to y we get:

$$v(x, y) = -\frac{x}{x^2 + y^2} + f(x)$$

Differentiating the above equation with respect to x and setting the result equal to the second of the Cauchy-Riemann equations, we get:

$$\begin{aligned}v_x &= \frac{x^2 - y^2}{(x^2 + y^2)^2} + f'(x) = \frac{x^2 - y^2}{(x^2 + y^2)^2} \\f'(x) &= 0 \\f(x) &= C\end{aligned}$$

Therefore, the harmonic conjugates of $u(x, y)$ are:

$$\boxed{v(x, y) = -\frac{x}{x^2 + y^2} + C}$$

3. Since v is a harmonic conjugate of u , from the Cauchy-Riemann equations we have:

$$u_x = v_y, \quad u_y = -v_x \tag{1}$$

Also, since u is a harmonic conjugate of v we have:

$$v_x = u_y, \quad v_y = -u_x \tag{2}$$

Taking the second of Equations (2) and substituting into the first of Equations (1) we get:

$$\begin{aligned}u_x &= v_y \\u_x &= -u_x \\2u_x &= 0 \\u_x &= 0\end{aligned}$$

Therefore, $u(x, y) = f(y)$. Similarly, by taking the first of Equations (2) and substituting into the second of Equations (1) we get $u(x, y) = g(x)$. In order for $f(y) = g(x)$, it must be the case that $u(x, y) = f(y) = g(x) = C$.

Since u is constant, from the Cauchy-Riemann equations we have $v_y = 0$ and $v_x = 0$. Thus, $v(x, y) = C$ as well.