

Math 417 – Section 28 Solutions

1. (b) Show that $\exp\left(\frac{2+i\pi}{4}\right) = \sqrt{\frac{e}{2}}(1+i)$.

$$\begin{aligned}\exp\left(\frac{2+i\pi}{4}\right) &= \exp\left(\frac{1}{2} + i\frac{\pi}{4}\right) \\ &= \exp\left(\frac{1}{2}\right) \exp\left(i\frac{\pi}{4}\right) \\ &= e^{1/2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) \\ &= \sqrt{e} \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) \\ &= \sqrt{\frac{e}{2}}(1+i)\end{aligned}$$

- (c) Show that $\exp(z+i\pi) = -\exp(z)$.

$$\begin{aligned}\exp(z+i\pi) &= \exp(z) \exp(i\pi) \\ &= \exp(z)(\cos \pi + i \sin \pi) \\ &= \exp(z)(-1 + i0) \\ &= -\exp(z)\end{aligned}$$

3. Consider the function $f(z) = \exp(\bar{z})$. Letting $z = x + iy$, we have:

$$\begin{aligned}f(z) &= \exp(\bar{z}) \\ &= \exp(x - iy) \\ &= e^x (\cos y - i \sin y)\end{aligned}$$

Therefore, $u(x, y) = e^x \cos y$ and $v(x, y) = -e^x \sin y$. The first partials are:

$$\begin{aligned}u_x &= e^x \cos y, & u_y &= -e^x \sin y \\ v_x &= -e^x \sin y, & v_y &= -e^x \cos y\end{aligned}$$

In order for the Cauchy-Riemann equations to be satisfied, it must be the case that:

$$\begin{aligned}u_x &= v_y &\Rightarrow & e^x \cos y = 0 \\ u_y &= -v_x &\Rightarrow & e^x \sin y = 0\end{aligned}$$

Since $e^x > 0$ for all x , it must be the case that $\cos y = 0$ and $\sin y = 0$ simultaneously. However, this cannot happen. Therefore, the Cauchy-Riemann equations can never be satisfied and, thus, $f(z)$ is not analytic anywhere.

5. First, we write $|\exp(2z+i)|$ and $|\exp(iz^2)|$ in terms of x and y by letting $z = x + iy$:

$$\begin{aligned}|\exp(2z+i)| &= |\exp(2(x+iy)+i)| \\ &= |\exp(2x+i(2y)+i)| \\ &= |\exp(2x+i(2y+1))| \\ &= e^{2x} \\ |\exp(iz^2)| &= |\exp(i(x+iy)^2)| \\ &= |\exp(i(x^2-y^2+i(2xy)))| \\ &= |\exp(-2xy+i(x^2-y^2))| \\ &= e^{-2xy}\end{aligned}$$

Therefore, using the Triangle Inequality we have:

$$|\exp(2z + i) + \exp(iz^2)| \leq |\exp(2z + i)| + |\exp(iz^2)| = e^{2x} + e^{-2xy}$$