

Math 417 – Sections 29 & 30 Solutions

1. (b) Show that $\text{Log}(1 - i) = \frac{1}{2} \ln 2 - i\frac{\pi}{4}$.

First, we find the modulus $r = |z|$ and principal argument Θ of $z = 1 - i$.

$$\begin{aligned} r = |z| &= \sqrt{1^2 + (-1)^2} = \sqrt{2} \\ \cos \Theta &= \frac{1}{\sqrt{2}}, \quad \sin \Theta = -\frac{1}{\sqrt{2}} \quad \Rightarrow \quad \Theta = -\frac{\pi}{4} \end{aligned}$$

The principal logarithm of z is then:

$$\begin{aligned} \text{Log}(1 - i) &= \ln r + i\Theta \\ &= \ln \sqrt{2} + i\left(-\frac{\pi}{4}\right) \\ &= \frac{1}{2} \ln 2 - i\frac{\pi}{4} \end{aligned}$$

2. (c) Verify that when $n = 0, \pm 1, \pm 2, \dots$,

$$\log(-1 + i\sqrt{3}) = \ln 2 + 2\left(n + \frac{1}{3}\right)\pi i$$

First, we find the modulus $r = |z|$ and principal argument Θ of $z = -1 + i\sqrt{3}$.

$$\begin{aligned} r = |z| &= \sqrt{(-1)^2 + (\sqrt{3})^2} = 2 \\ \cos \Theta &= -\frac{1}{2}, \quad \sin \Theta = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \Theta = \frac{2\pi}{3} \end{aligned}$$

The logarithm of z is then:

$$\begin{aligned} \log(1 - i) &= \ln r + i(\Theta + 2n\pi) \\ &= \ln 2 + i\left(\frac{2\pi}{3} + 2n\pi\right) \\ &= \ln 2 + 2\left(n + \frac{1}{3}\right)\pi i \end{aligned}$$

where $n = 0, \pm 1, \pm 2, \dots$

3. (a) To show that $\text{Log}(1 + i)^2 = 2\text{Log}(1 + i)$ we note that $(1 + i)^2 = 2i$. The modulus is 2 and the principal argument is $\frac{\pi}{2}$. Therefore, the principal logarithm is:

$$\begin{aligned} \text{Log}(1 + i)^2 &= \text{Log}(2i) \\ &= \ln 2 + i\frac{\pi}{2} \\ &= 2\left(\frac{1}{2} \ln 2 + i\frac{\pi}{4}\right) \\ &= 2\left(\ln \sqrt{2} + i\frac{\pi}{4}\right) \end{aligned}$$

Also note that the modulus of $1 + i$ is $\sqrt{2}$ and the principal argument is $\frac{\pi}{4}$. So its principal logarithm is:

$$\text{Log}(1 + i) = \ln \sqrt{2} + i\frac{\pi}{4}$$

Therefore,

$$\text{Log}(1 + i)^2 = 2\left(\ln \sqrt{2} + i\frac{\pi}{4}\right) = 2\text{Log}(1 + i)$$