

Math 417 – Section 32 Solutions

1. (a) Show that when $n = 0, \pm 1, \pm 2, \dots$

$$(1+i)^i = \exp\left(-\frac{\pi}{4} + 2n\pi\right) \exp\left(\frac{i}{2} \ln 2\right)$$

We begin by writing $(1+i)^i$ as follows:

$$(1+i)^i = \exp(i \log(1+i))$$

The modulus of $1+i$ is $\sqrt{2}$ and the principal argument is $\frac{\pi}{4}$. Therefore,

$$\begin{aligned} \log(1+i) &= \ln \sqrt{2} + i \left(\frac{\pi}{4} + 2n\pi\right) \\ \Rightarrow i \log(1+i) &= i \ln \sqrt{2} + i^2 \left(\frac{\pi}{4} + 2n\pi\right) \\ &= -\frac{\pi}{4} - 2n\pi + \frac{i}{2} \ln 2 \\ &= -\frac{\pi}{4} + 2n\pi + \frac{i}{2} \ln 2 \end{aligned}$$

Then we have:

$$\begin{aligned} (1+i)^i &= \exp(i \log(1+i)) \\ &= \exp\left(-\frac{\pi}{4} + 2n\pi + \frac{i}{2} \ln 2\right) \\ &= \exp\left(-\frac{\pi}{4} + 2n\pi\right) \exp\left(\frac{i}{2} \ln 2\right) \end{aligned}$$

where $n = 0, \pm 1, \pm 2, \dots$

2. (c) To find the principal value of $(1-i)^{4i}$ we start by writing:

$$(1-i)^{4i} = \exp(4i \operatorname{Log}(1-i))$$

The modulus of $1-i$ is $\sqrt{2}$ and the principal argument is $-\frac{\pi}{4}$. Therefore,

$$\begin{aligned} \operatorname{Log}(1-i) &= \ln \sqrt{2} + i \left(-\frac{\pi}{4}\right) \\ \Rightarrow 4i \operatorname{Log}(1-i) &= 4i \ln \sqrt{2} + 4i^2 \left(-\frac{\pi}{4}\right) \\ &= \pi + i(2 \ln 2) \end{aligned}$$

The principal value of $(1-i)^{4i}$ is then:

$$\begin{aligned} (1-i)^{4i} &= \exp(4i \operatorname{Log}(1-i)) \\ &= \exp(\pi + i(2 \ln 2)) \\ &= e^\pi (\cos(2 \ln 2) + i \sin(2 \ln 2)) \end{aligned}$$