

Math 417 – Section 33 Solutions

1. The derivatives of $\sin z$ and $\cos z$ are:

$$\begin{aligned}
 \frac{d}{dz} \sin z &= \frac{d}{dz} \left(\frac{e^{iz} - e^{-iz}}{2i} \right) \\
 &= \frac{ie^{iz} + ie^{-iz}}{2i} \\
 &= \frac{e^{iz} + e^{-iz}}{2} \\
 &= \cos z \\
 \frac{d}{dz} \cos z &= \frac{d}{dz} \left(\frac{e^{iz} + e^{-iz}}{2} \right) \\
 &= \frac{ie^{iz} - ie^{-iz}}{2} \\
 &= i \frac{e^{iz} - e^{-iz}}{2} \\
 &= -\frac{e^{iz} - e^{-iz}}{2i} \\
 &= -\sin z
 \end{aligned}$$

2. Using the definitions of $\cos z$ and $\sin z$ we have:

$$\begin{aligned}
 \cos z + i \sin z &= \frac{e^{iz} + e^{-iz}}{2} + i \left(\frac{e^{iz} - e^{-iz}}{2i} \right) \\
 &= \frac{1}{2}e^{iz} + \frac{1}{2}e^{-iz} + \frac{1}{2}e^{iz} - \frac{1}{2}e^{-iz} \\
 &= e^{iz}
 \end{aligned}$$

16. (a) Starting with $\overline{\cos(iz)}$ we have:

$$\begin{aligned}
 \overline{\cos(iz)} &= \overline{\cos(i(x + iy))} \\
 &= \overline{\cos(-y + ix)} \\
 &= \overline{\cos(-y) \cosh x - i \sin(-y) \sinh x} \\
 &= \overline{\cos y \cosh x + i \sin y \sinh x} \\
 &= \cos y \cosh x - i \sin y \sinh x
 \end{aligned}$$

using Equation (12) on p. 102. Then, noting that $i\bar{z} = i(x - iy) = y + ix$ we have:

$$\cos(i\bar{z}) = \cos(y + ix) = \cos y \cosh x - i \sin y \sinh x$$

Therefore,

$$\overline{\cos(iz)} = \cos(i\bar{z})$$

- (b) Starting with $\overline{\sin(iz)}$ we have:

$$\begin{aligned}
 \overline{\sin(iz)} &= \overline{\sin(i(x + iy))} \\
 &= \overline{\sin(-y + ix)} \\
 &= \overline{\sin(-y) \cosh x + i \cos(-y) \sinh x} \\
 &= \overline{-\sin y \cosh x + i \cos y \sinh x} \\
 &= -\sin y \cosh x - i \cos y \sinh x
 \end{aligned}$$

using Equation (11) on p. 102. Then, noting that $i\bar{z} = i(x - iy) = y + ix$ we have:

$$\sin(i\bar{z}) = \sin(y + ix) = \sin y \cosh x + i \cos y \sinh x$$

The expressions $\overline{\sin(iz)}$ and $\sin(i\bar{z})$ are equal when:

$$\sin y \cosh x = 0$$

$$\cos y \sinh x = 0$$

Since $\cosh x \geq 0$ for all x , it must be the case that $\sin y = 0$. The solutions to the latter equation are $y = n\pi$ where $n = 0, \pm 1, \pm 2, \dots$

When $y = n\pi$, $\cos y$ is either 1 or -1 . In either case, the second of the above equations tells us that $\sinh x = 0$. This occurs exactly when $x = 0$. Therefore, $\overline{\sin(iz)} = \sin(i\bar{z})$ when:

$$z = 0 + i(n\pi) = in\pi, \quad n = 0, \pm 1, \pm 2, \dots$$