

Math 417 – Section 38 Solutions

2. Starting with $z(\theta) = 2e^{i\theta} = 2(\cos \theta + i \sin \theta)$ and plugging in:

$$\theta = \phi(y) = \arctan \frac{y}{\sqrt{4-y^2}}$$

we have:

$$\begin{aligned} z(y) &= 2 \left[\cos \left(\arctan \frac{y}{\sqrt{4-y^2}} \right) + i \sin \left(\arctan \frac{y}{\sqrt{4-y^2}} \right) \right] \\ &= 2 \left[\frac{\sqrt{4-y^2}}{2} + i \frac{y}{2} \right] \\ &= \sqrt{4-y^2} + iy \end{aligned}$$

Taking the derivative of $\phi(y)$ we have:

$$\begin{aligned} \phi'(y) &= \frac{1}{1 + \left(\frac{y}{\sqrt{4-y^2}}\right)^2} \frac{d}{dy} \left(\frac{y}{\sqrt{4-y^2}} \right) \\ &= \left(\frac{1}{1 + \frac{y^2}{4-y^2}} \right) \left(\frac{\sqrt{4-y^2} - y \cdot \frac{1}{2}(4-y^2)^{-1/2}(-2y)}{4-y^2} \right) \\ &= \left(\frac{4-y^2}{4-y^2+y^2} \right) \left(\frac{4-y^2+y^2}{(4-y^2)^{3/2}} \right) \\ &= \left(\frac{4-y^2}{4} \right) \left(\frac{4}{(4-y^2)^{3/2}} \right) \\ &= \frac{1}{\sqrt{4-y^2}} \end{aligned}$$

Note that $\phi'(y) > 0$ for $-2 < y < 2$.