

# Math 417 – Sections 39 & 40 Solutions

1. Consider the function  $f(z) = (z + 2)/z$ . Evaluate  $\int_C f(z) dz$  over the given contour  $C$ .

(a)  $C$  is the semicircle  $z = 2e^{i\theta}$  where  $0 \leq \theta \leq \pi$ .

$$\begin{aligned} \int_C f(z) dz &= \int_0^\pi f(z(\theta)) z'(\theta) d\theta \\ &= \int_0^\pi \left( \frac{2e^{i\theta} + 2}{2e^{i\theta}} \right) (2ie^{i\theta}) d\theta \\ &= 2 \int_0^\pi (ie^{i\theta} + i) d\theta \\ &= 2 [e^{i\theta} + i\theta]_0^\pi \\ &= 2 [(e^{i\pi} + i\pi) - (e^0 + 0)] \\ &= -4 + 2\pi i \end{aligned}$$

(b)  $C$  is the semicircle  $z = 2e^{i\theta}$  where  $\pi \leq \theta \leq 2\pi$ .

$$\begin{aligned} \int_C f(z) dz &= \int_\pi^{2\pi} f(z(\theta)) z'(\theta) d\theta \\ &= \int_\pi^{2\pi} \left( \frac{2e^{i\theta} + 2}{2e^{i\theta}} \right) (2ie^{i\theta}) d\theta \\ &= 2 \int_\pi^{2\pi} (ie^{i\theta} + i) d\theta \\ &= 2 [e^{i\theta} + i\theta]_\pi^{2\pi} \\ &= 2 [(e^{i2\pi} + i2\pi) - (e^{i\pi} + i\pi)] \\ &= 4 + 2\pi i \end{aligned}$$

(c)  $C$  is the circle  $z = 2e^{i\theta}$  where  $0 \leq \theta \leq 2\pi$

$$\int_C f(z) dz = (-4 + 2\pi i) + (4 + 2\pi i) = 4\pi i$$

2. Consider the function  $f(z) = z - 1$ . Evaluate  $\int_C f(z) dz$  over the given contour  $C$ .

(a)  $C$  is the semicircle  $z = 1 + e^{i\theta}$  where  $\pi \leq \theta \leq 2\pi$ .

$$\begin{aligned} \int_C f(z) dz &= \int_\pi^{2\pi} f(z(\theta)) z'(\theta) d\theta \\ &= \int_\pi^{2\pi} (e^{i\theta}) (ie^{i\theta}) d\theta \\ &= \int_\pi^{2\pi} ie^{2i\theta} d\theta \\ &= \left[ \frac{1}{2} e^{2i\theta} \right]_\pi^{2\pi} \\ &= \frac{1}{2} [e^{4\pi i} - e^{2\pi i}] \\ &= 0 \end{aligned}$$

(b)  $C$  is the segment  $0 \leq x \leq 2$  of the real axis. In this case, we let  $z(x) = x + i0$  where  $0 \leq x \leq 2$ .

$$\begin{aligned}
 \int_C f(z) dz &= \int_0^2 f(z(x)) z'(x) dx \\
 &= \int_0^2 (x-1) dx \\
 &= \left[ \frac{1}{2}x^2 - x \right]_0^2 \\
 &= \left[ \frac{1}{2}(2)^2 - 2 \right] - \left[ \frac{1}{2}(0)^2 - 0 \right] \\
 &= 0
 \end{aligned}$$

3. We will integrate  $f(z) = \pi \exp(\pi \bar{z})$  along each side of the square.

I. The contour  $C_I$  from  $z = 0$  to  $z = 1$  can be described by  $z(t) = t$ ,  $0 \leq t \leq 1$ . The integral is then:

$$\begin{aligned}
 \int_{C_I} f(z) dz &= \int_0^1 f(z(t)) z'(t) dt \\
 &= \int_0^1 \pi \exp(\pi t) dt \\
 &= [\exp(\pi t)]_0^1 \\
 &= e^\pi - 1
 \end{aligned}$$

II. The contour  $C_{II}$  from  $z = 1$  to  $z = 1 + i$  can be described by  $z(t) = 1 + it$ ,  $0 \leq t \leq 1$ . The integral is then:

$$\begin{aligned}
 \int_{C_{II}} f(z) dz &= \int_0^1 f(z(t)) z'(t) dt \\
 &= \int_0^1 \pi \exp(\pi(1 - it)) i dt \\
 &= -[\exp(\pi(1 - it))]_0^1 \\
 &= -[\exp(\pi(1 - i)) - \exp(\pi)] \\
 &= 2e^\pi
 \end{aligned}$$

III. The contour  $C_{III}$  from  $z = 1 + i$  to  $z = i$  can be described by  $z(t) = t + i$ ,  $0 \leq t \leq 1$ . The integral is then:

$$\begin{aligned}
 \int_{C_{III}} f(z) dz &= \int_1^0 f(z(t)) z'(t) dt \\
 &= \int_1^0 \pi \exp(\pi(t - i)) dt \\
 &= [\exp(\pi(t - i))]_1^0 \\
 &= \exp(-\pi i) - \exp(\pi(1 - i)) \\
 &= -1 + e^\pi
 \end{aligned}$$

IV. The contour  $C_{IV}$  from  $z = i$  to  $z = 0$  can be described by  $z(t) = it$ ,  $0 \leq t \leq 1$ . The integral is

then:

$$\begin{aligned}
\int_{C_{IV}} f(z) dz &= \int_1^0 f(z(t)) z'(t) dt \\
&= \int_1^0 \pi \exp(-\pi it) i dt \\
&= -[\exp(-\pi it)]_1^0 \\
&= -[\exp(0) - \exp(-\pi i)] \\
&= -2
\end{aligned}$$

The integral over  $C$  is then:

$$\int_C f(z) dz = e^\pi - 1 + 2e^\pi - 1 + e^\pi - 2 = 4e^\pi - 4$$

8. Using the representation  $z = \sqrt{4-y^2} + iy$ ,  $-2 \leq y \leq 2$  for  $C$  to compute  $\int_C \bar{z} dz$ , we have:

$$\begin{aligned}
\int_C \bar{z} dz &= \int_{-2}^2 (\sqrt{4-y^2} - iy) z'(y) dy \\
&= \int_{-2}^2 (\sqrt{4-y^2} - iy) \left( -\frac{y}{\sqrt{4-y^2}} + i \right) dy \\
&= \int_{-2}^2 \left[ (-y + y) + i \left( \frac{y^2}{\sqrt{4-y^2}} + \sqrt{4-y^2} \right) \right] dy \\
&= i \int_{-2}^2 \frac{4}{\sqrt{4-y^2}} dy \\
&= i \left[ 4 \sin^{-1} \frac{y}{2} \right]_{-2}^2 \\
&= i \left[ 4 \left( \frac{\pi}{2} \right) - 4 \left( -\frac{\pi}{2} \right) \right] \\
&= 4\pi i
\end{aligned}$$