

Math 417 – Sections 47 & 48 Solutions

1. (a) The function $f(z) = e^{-z}$ is analytic everywhere inside and on C . Using the Cauchy Integral Formula, we have:

$$\int_C \frac{e^{-z} dz}{z - (\pi i/2)} = 2\pi i f\left(\frac{\pi i}{2}\right) = 2\pi i e^{-\pi i/2} = 2\pi i \left[\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right) \right] = 2\pi$$

- (b) The function $f(z) = \cos z/(z^2 + 8)$ is analytic everywhere inside and on C (it is singular at $z = \pm 2i\sqrt{2}$ which lie outside of C). Using the Cauchy Integral Formula, we have:

$$\int_C \frac{\cos z}{z(z^2 + 8)} dz = 2\pi i f(0) = 2\pi i \left(\frac{\cos 0}{0^2 + 8} \right) = \frac{\pi i}{4}$$

- (c) The function $f(z) = z/2$ is analytic everywhere inside and on C . Using the Cauchy Integral Formula, we have:

$$\int_C \frac{z}{2z + 1} dz = \int_C \frac{z/2}{z + 1/2} dz = 2\pi i f\left(-\frac{1}{2}\right) = 2\pi i \left(\frac{-1/2}{2} \right) = -\frac{\pi i}{2}$$

2. (a) The function $g(z) = 1/(z^2 + 4)$ is singular at $z = \pm 2i$. The point $z = 2i$ is inside C . Let $f(z) = 1/(z + 2i)$ and note that $f(z)$ is analytic everywhere on and inside C . Using the Cauchy Integral Formula, we have:

$$\int_C \frac{dz}{z^2 + 4} = \int_C \frac{1/(z + 2i)}{z - 2i} dz = 2\pi i f(2i) = 2\pi i \left(\frac{1}{2i + 2i} \right) = \frac{\pi}{2}$$

- (b) Here, we let $f(z) = 1/(z + 2i)^2$ and note that $f'(z) = -2/(z + 2i)^3$. Using the Cauchy Integral Formula, we have:

$$\int_C \frac{dz}{(z^2 + 4)^2} = \int_C \frac{1/(z + 2i)^2}{(z - 2i)^2} dz = \frac{2\pi i}{1!} f'(2i) = 2\pi i \left(\frac{-2}{(2i + 2i)^3} \right) = \frac{\pi}{16}$$

4. When $z = w$ is outside of C , the integrand is analytic everywhere on and inside C . By the Cauchy-Goursat Theorem, the integral is 0 in this case.

When $z = w$ is inside C , the integrand is singular at $z = w$. Letting $f(z) = z^3 + 2z$, we have $f'(z) = 3z^2 + 2$ and $f''(z) = 6z$. Using the Cauchy Integral Formula, we have:

$$\int_C \frac{z^3 + 2z}{(z - w)^3} dz = \frac{2\pi i}{2!} f''(w) = 6\pi i w$$