

Math 417 – Section 5 Solutions

1. (a) $\overline{\overline{z} + 3i} = \overline{\overline{z}} + \overline{3i} = z - 3i$
 (b) $\overline{i\overline{z}} = \overline{i}\overline{\overline{z}} = -i\overline{z}$
 (c) $\overline{(2+i)^2} = (\overline{2+i})^2 = (2-i)^2 = 4 - 4i + i^2 = 3 - 4i$
 (d) $|(2\overline{z} + 5)(\sqrt{2} - i)| = |2\overline{z} + 5||\sqrt{2} - i| = |\overline{2z + 5}|\sqrt{3} = |2z + 5|\sqrt{3}$

3. First, we let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then we have:

$$\begin{aligned}\overline{z_1 - z_2} &= \overline{(x_1 + iy_1) - (x_2 + iy_2)} \\ &= \overline{(x_1 - x_2) + i(y_1 - y_2)} \\ &= (x_1 - x_2) - i(y_1 - y_2) \\ &= (x_1 - iy_1) - (x_2 - iy_2) \\ &= \overline{z_1} - \overline{z_2}\end{aligned}$$

and

$$\begin{aligned}\overline{z_1 z_2} &= \overline{(x_1 + iy_1)(x_2 + iy_2)} \\ &= \overline{(x_1 x_2 - y_1 y_2) + i(y_2 x_1 + x_2 y_1)} \\ &= (x_1 x_2 - y_1 y_2) - i(y_2 x_1 + x_2 y_1) \\ &= (x_1 - iy_1)(x_2 - iy_2) \\ &= \overline{z_1} \overline{z_2}\end{aligned}$$

16. (a) In the first step, we have:

$$\begin{aligned}|z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} \\ &= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\ &= z_1 \overline{z_1} + z_1 \overline{z_2} + z_2 \overline{z_1} + z_2 \overline{z_2} \\ &= z_1 \overline{z_1} + (z_1 \overline{z_2} + \overline{z_1 z_2}) + z_2 \overline{z_2}\end{aligned}$$

- (b) Next, we use the fact that:

$$\operatorname{Re}(z) = \frac{z + \overline{z}}{2} \quad \text{or} \quad z + \overline{z} = 2\operatorname{Re}(z)$$

to say that:

$$z_1 \overline{z_2} + \overline{z_1 z_2} = 2\operatorname{Re}(z_1 \overline{z_2})$$

Then we use the fact that:

$$\operatorname{Re}(z) \leq |\operatorname{Re}(z)| \leq |z|$$

to say that:

$$z_1 \overline{z_2} + \overline{z_1 z_2} = 2\operatorname{Re}(z_1 \overline{z_2}) \leq 2|z_1 \overline{z_2}| = 2|z_1||\overline{z_2}| = 2|z_1||z_2|$$

- (c) Finally, to prove the Triangle Inequality we use the results from (a) and (b) as follows:

$$\begin{aligned}|z_1 + z_2|^2 &\leq z_1 \overline{z_1} + 2|z_1||z_2| + z_2 \overline{z_2} \\ |z_1 + z_2|^2 &\leq |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \\ |z_1 + z_2|^2 &\leq (|z_1| + |z_2|)^2 \\ |z_1 + z_2| &\leq |z_1| + |z_2|\end{aligned}$$