

Math 417 – Sections 55 & 56 Solutions

1. In the domain $0 < |z| < \infty$, the function $f(z) = z^2 \sin\left(\frac{1}{z^2}\right)$ has the Laurent Series:

$$\begin{aligned} f(z) &= z^2 \left[\frac{1}{z^2} - \frac{1}{3!} \cdot \frac{1}{z^6} + \frac{1}{5!} \cdot \frac{1}{z^{10}} - \frac{1}{7!} \cdot \frac{1}{z^{14}} + \dots \right] \\ &= 1 - \frac{1}{3!} \cdot \frac{1}{z^4} + \frac{1}{5!} \cdot \frac{1}{z^8} - \frac{1}{7!} \cdot \frac{1}{z^{12}} + \dots \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{1}{z^{4n}} \end{aligned}$$

2. To derive the Laurent Series representation of $f(z) = \frac{e^z}{(z+1)^2}$ about $z = -1$, we first note that the function e^z has the following Taylor Series representation about $z = -1$ for all z :

$$e^z = \frac{1}{e} \cdot e^{z+1} = \frac{1}{e} \left[1 + (z+1) + \frac{(z+1)^2}{2!} + \frac{(z+1)^3}{3!} + \dots \right]$$

The Laurent Series for $f(z)$ about $z = -1$ is then:

$$\begin{aligned} f(z) &= \frac{e^z}{(z+1)^2} \\ &= \frac{1}{e} \cdot \frac{1}{(z+1)^2} \left[1 + (z+1) + \frac{(z+1)^2}{2!} + \frac{(z+1)^3}{3!} + \dots \right] \\ &= \frac{1}{e} \left[\frac{1}{(z+1)^2} + \frac{1}{z+1} + \frac{1}{2!} + \frac{z+1}{3!} + \dots \right] \\ &= \frac{1}{e} \left[\frac{1}{(z+1)^2} + \frac{1}{z+1} + \sum_{n=0}^{\infty} \frac{(z+1)^n}{(n+2)!} \right] \end{aligned}$$

for $0 < |z+1| < \infty$.

4. The function $f(z) = \frac{1}{z^2(1-z)}$ has singular points at $z = 0$ and $z = 1$. Therefore, the function has a Laurent Series representation about $z = 0$ in the region $0 < |z| < 1$ and in the region $1 < |z| < \infty$.

First, in the region $0 < |z| < 1$, we note that $\frac{1}{1-z}$ has a Maclaurin Series representation. Therefore, the Laurent Series for $f(z)$ is:

$$\begin{aligned} f(z) &= \frac{1}{z^2} \cdot \frac{1}{1-z} \\ &= \frac{1}{z^2} (1 + z + z^2 + z^3 + \dots) \\ &= \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \dots \\ &= \frac{1}{z^2} + \frac{1}{z} + \sum_{n=0}^{\infty} z^n \end{aligned}$$

Next, in the region $1 < |z| < \infty$, we have:

$$\begin{aligned}
 f(z) &= \frac{1}{z^2} \cdot \frac{1}{1-z} \\
 &= \frac{1}{z^2} \cdot \frac{1/z}{(1/z)-1} \\
 &= -\frac{1}{z^3} \cdot \frac{1}{1-(1/z)} \\
 &= -\frac{1}{z^3} \left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots \right) \\
 &= -\frac{1}{z^3} - \frac{1}{z^4} - \frac{1}{z^5} - \dots \\
 &= -\sum_{n=3}^{\infty} \frac{1}{z^n}
 \end{aligned}$$

5. (a) The Maclaurin Series for $f(z) = \frac{z+1}{z-1}$:

$$\begin{aligned}
 f(z) &= \frac{z+1}{z-1} \\
 &= -(z+1) \cdot \frac{1}{1-z} \\
 &= -(z+1)(1+z+z^2+z^3+\dots) \\
 &= -[(1+z+z^2+z^3+\dots) + (z+z^2+z^3+\dots)] \\
 &= -(1+2z+2z^2+2z^3+\dots) \\
 &= -1 - 2 \sum_{n=1}^{\infty} z^n
 \end{aligned}$$

is valid in the region $|z| < 1$.

- (b) The Laurent Series of $f(z)$ in the domain $1 < |z| < \infty$ is:

$$\begin{aligned}
 f(z) &= (z+1) \cdot \frac{1}{z-1} \\
 &= (z+1) \cdot \frac{1/z}{1-(1/z)} \\
 &= \frac{z+1}{z} \left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots \right) \\
 &= (z+1) \left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots \right) \\
 &= \left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots \right) + \left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots \right) \\
 &= 1 + \frac{2}{z} + \frac{2}{z^2} + \frac{2}{z^3} + \dots \\
 &= 1 + 2 \sum_{n=1}^{\infty} \frac{1}{z^n}
 \end{aligned}$$