Math 417 – Sections 57–59 Solutions

1. Differentiating the Maclaurin Series $f(z) = \dfrac{1}{1-z} = \displaystyle\sum_{n=0}^{\infty} z^n$ we have:

$$\begin{aligned}
\frac{d}{dz}\frac{1}{1-z} &= \frac{d}{dz}\sum_{n=0}^{\infty} z^n \\
\frac{1}{(1-z)^2} &= \sum_{n=1}^{\infty} nz^{n-1} \\
&= \sum_{n=0}^{\infty}(n+1)z^n
\end{aligned}$$

Differentiating the above series, we have:

$$\begin{aligned}
\frac{d}{dz}\frac{1}{1-z)^2} &= \frac{d}{dz}\sum_{n=0}^{\infty}(n+1)z^n \\
\frac{2}{(1-z)^3} &= \sum_{n=1}^{\infty} n(n+1)z^{n-1} \\
&= \sum_{n=0}^{\infty}(n+1)(n+2)z^n
\end{aligned}$$

3. The Taylor Series of the function

$$\frac{1}{z} = \frac{1}{2+(z-2)} = \frac{1}{2}\cdot\frac{1}{1+(z-2)/2}$$

about $z = 2$ is:

$$\frac{1}{z} = \frac{1}{2}\left[1 - \frac{z-2}{2} + \frac{(z-2)^2}{2^2} - \frac{(z-2)^3}{2^3} + \ldots\right]$$

Differentiating the above series we have:

$$\begin{aligned}
\frac{d}{dz}\frac{1}{z} &= \frac{d}{dz}\frac{1}{2}\left[1 - \frac{z-2}{2} + \frac{(z-2)^2}{2^2} - \frac{(z-2)^3}{2^3} + \ldots\right] \\
-\frac{1}{z^2} &= \frac{1}{2}\left[-1\cdot\frac{1}{2} + 2\cdot\frac{z-2}{2^2} - 3\cdot\frac{(z-2)^2}{2^3} + \ldots\right] \\
\frac{1}{z^2} &= 1\cdot\frac{1}{2^2} - 2\cdot\frac{z-2}{2^3} + 3\cdot\frac{(z-2)^2}{2^4} - \ldots \\
&= \sum_{n=0}^{\infty}(-1)^n(n+1)\cdot\frac{(z-2)^n}{2^{n+2}} \\
&= \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n
\end{aligned}$$

Since $\frac{1}{z}$ is singular at $z = 0$, the Taylor Series is valid for $|z-2| < 2$.