

Math 417 – Section 61 Solutions

1. The Laurent Series for $f(z) = \frac{e^z}{z(z^2 + 1)}$ in the region $0 < |z| < 1$ is:

$$\begin{aligned} f(z) &= \frac{1}{z} \cdot e^z \cdot \frac{1}{z^2 + 1} \\ &= \frac{1}{z} \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \right) (1 - z^2 + z^4 - z^6 + \dots) \\ &= \frac{1}{z} \left[1 + z + \left(-1 + \frac{1}{2!} \right) z^2 + \left(-1 + \frac{1}{3!} \right) z^3 + \dots \right] \\ &= \frac{1}{z} + 1 - \frac{1}{2}z - \frac{5}{6}z^2 + \dots \end{aligned}$$

3. The Laurent Series for $f(z) = \frac{1}{e^z - 1}$ in the region $0 < |z| < 2\pi$ is:

$$\begin{aligned} f(z) &= \frac{1}{e^z - 1} \\ &= \frac{1}{(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots) - 1} \\ &= \frac{1}{z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots} \end{aligned}$$

The long division goes as follows:

$$\begin{array}{r} \frac{1}{z} - \frac{1}{2} + \frac{z}{12} - \frac{z^3}{720} + \dots \\ z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \dots \overline{) 1} \\ 1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \frac{z^4}{120} + \dots \\ \hline -\frac{z}{2} - \frac{z^2}{6} - \frac{z^3}{24} - \frac{z^4}{120} + \dots \\ -\frac{z}{2} - \frac{z^2}{4} - \frac{z^3}{12} - \frac{z^4}{48} + \dots \\ \hline \frac{z^2}{12} + \frac{z^3}{24} + \frac{z^4}{80} + \dots \\ \frac{z^2}{12} + \frac{z^3}{24} + \frac{z^4}{72} + \dots \\ \hline -\frac{z^4}{720} + \dots \\ -\frac{z^4}{720} + \dots \\ \hline 0 + \dots \end{array}$$

5. (a) The multiplication has already been performed so that we have:

$$(d_0 - 1) + d_1 z + \left(d_2 + \frac{1}{3!}d_0\right) z^2 + \left(d_3 + \frac{1}{3!}d_1\right) z^3 + \left(d_4 + \frac{1}{3!}d_2 + \frac{1}{5!}d_0\right) + \dots = 0$$

- (b) Setting the coefficients of z^n on the left hand side of the above equation equal to 0, we have:

$$\begin{aligned} d_0 - 1 &= 0 \\ d_1 &= 0 \\ d_2 + \frac{1}{3!}d_0 &= 0 \\ d_3 + \frac{1}{3!}d_1 &= 0 \\ d_4 + \frac{1}{3!}d_2 + \frac{1}{5!}d_0 &= 0 \\ &\vdots \end{aligned}$$

Therefore, we have:

$$\begin{aligned} d_0 &= 1 \\ d_1 &= 0 \\ d_2 &= -\frac{1}{6} \\ d_3 &= 0 \\ d_4 &= \frac{7}{360} \\ &\vdots \end{aligned}$$