

Math 417 – Sections 62-64 Solutions

1. (a) The Laurent Series of $f(z) = \frac{1}{z+z^2}$ about $z=0$ is:

$$\begin{aligned} f(z) &= \frac{1}{z+z^2} \\ &= \frac{1}{z} \cdot \frac{1}{1+z} \\ &= \frac{1}{z}(1 - z + z^2 - z^3 + \dots) \\ &= \frac{1}{z} - 1 + z - z^2 + \dots \end{aligned}$$

The residue is $c_{-1} = 1$.

- (b) The Laurent Series of $f(z) = z \cos\left(\frac{1}{z}\right)$ about $z=0$ is:

$$\begin{aligned} f(z) &= z \cos\left(\frac{1}{z}\right) \\ &= z \left[1 - \frac{1}{2!} \cdot \left(\frac{1}{z}\right)^2 + \frac{1}{4!} \cdot \left(\frac{1}{z}\right)^4 - \dots \right] \\ &= z - \frac{1}{2!} \cdot \frac{1}{z} + \frac{1}{4!} \cdot \frac{1}{z^3} - \dots \end{aligned}$$

The residue is $c_{-1} = -\frac{1}{2!} = -\frac{1}{2}$.

- (c) The Laurent Series of $f(z) = \frac{z - \sin z}{z}$ about $z=0$ is:

$$\begin{aligned} f(z) &= \frac{z - \sin z}{z} \\ &= \frac{1}{z} \left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right) \right] \\ &= \frac{1}{z} \left(\frac{z^3}{3!} - \frac{z^5}{5!} + \dots \right) \\ &= \frac{z^2}{3!} - \frac{z^4}{5!} + \dots \end{aligned}$$

The residue is $c_{-1} = 0$.

2. (a) The function $f(z) = \frac{\exp(-z)}{z^2}$ has a singularity at $z=0$ and it lies inside the contour C : $|z|=3$.

The Laurent Series of $f(z)$ about $z=0$ is:

$$\begin{aligned} f(z) &= \frac{1}{z^2} \exp(-z) \\ &= \frac{1}{z^2} \left(1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \dots \right) \\ &= \frac{1}{z^2} - \frac{1}{z} + \frac{1}{2!} - \frac{z}{3!} + \dots \end{aligned}$$

The residue is $c_{-1} = -1$. Therefore,

$$\int_C f(z) dz = 2\pi i c_{-1} = -2\pi i$$

- (b) The function $f(z) = \frac{\exp(-z)}{(z-1)^2}$ has a singularity at $z = 1$ and it lies inside the contour C : $|z| = 3$.

The Laurent Series of $f(z)$ about $z = 1$ is:

$$\begin{aligned} f(z) &= \frac{1}{(z-1)^2} \exp(-z) \\ &= \frac{1}{(z-1)^2} \exp(-z+1-1) \\ &= \frac{\exp(-1)}{(z-1)^2} \exp(-(z-1)) \\ &= \frac{\exp(-1)}{(z-1)^2} \left[1 - (z-1) + \frac{(z-1)^2}{2!} - \frac{(z-1)^3}{3!} + \dots \right] \\ &= \frac{1}{e(z-1)^2} - \frac{1}{e(z-1)} + \frac{1}{2!e} - \frac{(z-1)}{3!e} + \dots \end{aligned}$$

The residue is $c_{-1} = -1/e$. Therefore,

$$\int_C f(z) dz = 2\pi i c_{-1} = -\frac{2\pi i}{e}$$

3. (a) The function $f(z) = \frac{z^5}{1-z^3}$ has singularities at $z = 1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ all of which lie inside the contour C : $|z| = 2$. Since none of these lies outside of C , we can compute the integral of $f(z)$ along C using:

$$\int_C f(z) dz = 2\pi i \operatorname{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right) \right]$$

The Laurent Series of $\frac{1}{z^2} f\left(\frac{1}{z}\right)$ about $z = 0$ is:

$$\begin{aligned} \frac{1}{z^2} f\left(\frac{1}{z}\right) &= \frac{1}{z^2} \cdot \frac{1/z^5}{1-1/z^3} \\ &= \frac{1}{z^2} \cdot \frac{1}{z^5 - z^2} \\ &= -\frac{1}{z^4} \cdot \frac{1}{1-z^3} \\ &= -\frac{1}{z^4} (1 + z^3 + z^6 + \dots) \\ &= -\frac{1}{z^4} - \frac{1}{z} - z^2 - \dots \end{aligned}$$

The residue is $c_{-1} = -1$. Therefore,

$$\int_C f(z) dz = 2\pi i c_{-1} = -2\pi i$$