

Math 417 – Section 65 Solutions

1. (a) The Laurent Series of $f(z) = z \exp\left(\frac{1}{z}\right)$ about $z = 0$ is:

$$\begin{aligned} f(z) &= z \exp\left(\frac{1}{z}\right) \\ &= z \left(1 + \frac{1}{z} + \frac{1}{2!} \cdot \frac{1}{z^2} + \frac{1}{3!} \cdot \frac{1}{z^3} \cdots\right) \\ &= z + 1 + \frac{1}{2!} \cdot \frac{1}{z} + \frac{1}{3!} \cdot \frac{1}{z^2} + \cdots \end{aligned}$$

Since there are infinite terms in the principal part of the Laurent Series, the point $z = 0$ is an essential singularity.

- (b) The Laurent Series of $f(z) = \frac{z^2}{1+z}$ about $z = -1$ is:

$$\begin{aligned} f(z) &= \frac{z^2}{1+z} \\ &= \frac{1}{1+z} \cdot z^2 \\ &= \frac{1}{1+z} [1 - 2(1+z) + (1+z)^2] \\ &= \frac{1}{1+z} - 2 + (1+z) \end{aligned}$$

Since the only term in the principal part is the $(z+1)^{-1}$ term, $z = -1$ is a simple pole.

- (d) The Laurent Series of $f(z) = \frac{\cos z}{z}$ about $z = 0$ is:

$$\begin{aligned} f(z) &= \frac{\cos z}{z} \\ &= \frac{1}{z} \cdot \cos z \\ &= \frac{1}{z} \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\right) \\ &= \frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} - \cdots \end{aligned}$$

Since the only term in the principal part is the z^{-1} term, $z = 0$ is a simple pole.

2. (b) The Laurent Series of $f(z) = \frac{1 - \exp(2z)}{z^4}$ about $z = 0$ is:

$$\begin{aligned} f(z) &= \frac{1 - \exp(2z)}{z^4} \\ &= \frac{1}{z^4} \left[1 - \left(1 + 2z + \frac{(2z)^2}{2!} + \frac{(2z)^3}{3!} + \cdots\right)\right] \\ &= \frac{1}{z^4} \left(-2z - 2z^2 - \frac{4z^3}{3} - \cdots\right) \\ &= -\frac{2}{z^3} - \frac{2}{z^2} - \frac{4}{3z} - \cdots \end{aligned}$$

The point $z = 0$ is a pole of order 3 and the residue is $c_{-1} = -\frac{4}{3}$.