

Math 417 – Sections 66 & 67 Solutions

1. (a) For the function $f(z) = \frac{z^2 + 2}{z - 1}$, let $\phi(z) = z^2 + 2$. Then,

$$f(z) = \frac{\phi(z)}{z - 1}$$

Note that $\phi(z)$ is analytic at $z = 1$ and is nonzero there. Therefore, $z = 1$ is a simple pole. The residue is:

$$\operatorname{Res}_{z=1} f(z) = \phi(1) = 1^2 + 2 = 3$$

- (b) For the function $f(z) = \left(\frac{z}{2z + 1}\right)^3$, let $\phi(z) = \frac{1}{8}z^3$. Then,

$$f(z) = \frac{\phi(z)}{(z + \frac{1}{2})^3}$$

Note that $\phi(z)$ is analytic at $z = -\frac{1}{2}$ and is nonzero there. Therefore, $z = -\frac{1}{2}$ is a pole of order 3. The residue is:

$$\operatorname{Res}_{z=-\frac{1}{2}} f(z) = \frac{\phi''(-\frac{1}{2})}{2!} = \frac{\frac{3}{4}(-\frac{1}{2})}{2!} = -\frac{3}{16}$$

2. (a) Using the branch cut $|z| > 0$, $0 < \arg z < 2\pi$, the residue is:

$$\operatorname{Res}_{z=-1} \frac{z^{1/4}}{z + 1} = (-1)^{1/4} = \exp\left(i\frac{\pi}{4}\right) = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1+i}{\sqrt{2}}$$

where we used the fact that $\arg(-1) = \pi$.

3. (a) Using the circle $|z - 2| = 2$ as the contour, the integral is:

$$\int_C \frac{3z^3 + 2}{(z - 1)(z^2 + 9)} dz = 2\pi i \operatorname{Res}_{z=1} f(z)$$

since $z = 1$ is the only singularity inside the circle. Letting $f(z) = \phi(z) = \frac{3z^3 + 2}{z^2 + 9}$ and noting that $z = 1$ is a simple pole, the residue is:

$$\operatorname{Res}_{z=1} f(z) = \phi(1) = \frac{3(1)^3 + 2}{1^2 + 9} = \frac{1}{2}$$

The value of the integral is then:

$$\int_C \frac{3z^3 + 2}{(z - 1)(z^2 + 9)} dz = \pi i$$

- (b) Using the circle $|z| = 4$, the integral is:

$$\int_C \frac{3z^3 + 2}{(z - 1)(z^2 + 9)} dz = 2\pi i \left(\operatorname{Res}_{z=1} f(z) + \operatorname{Res}_{z=3i} f(z) + \operatorname{Res}_{z=-3i} f(z) \right)$$

since all three singular points of $f(z)$ lie inside the contour. We've already computed the first residue in part (a). The other residues are:

$$\begin{aligned} \operatorname{Res}_{z=3i} f(z) &= \left. \frac{3z^3 + 2}{(z - 1)(z + 3i)} \right|_{z=3i} = \frac{3(3i)^3 + 2}{(3i - 1)(3i + 3i)} = \frac{-81i + 2}{-6i - 18} \\ \operatorname{Res}_{z=-3i} f(z) &= \left. \frac{3z^3 + 2}{(z - 1)(z - 3i)} \right|_{z=-3i} = \frac{3(-3i)^3 + 2}{(-3i - 1)(-3i - 3i)} = \frac{81i + 2}{6i - 18} \end{aligned}$$

The value of the integral is then:

$$\int_C \frac{3z^3 + 2}{(z-1)(z^2+9)} dz = 2\pi i \left(\frac{1}{2} + \frac{-81i+2}{-6i-18} + \frac{81i+2}{6i-18} \right) = 6\pi i$$

4. (a) Using the circle $|z| = 2$ as the contour, the integral is:

$$\int_C \frac{dz}{z^3(z+4)} = 2\pi i \operatorname{Res}_{z=0} f(z)$$

since $z = 0$ is the only singularity inside the circle. Letting $f(z) = \phi(z) = \frac{1}{z+4}$ and noting that $z = 0$ is a pole of order 3, the residue is:

$$\operatorname{Res}_{z=0} f(z) = \frac{\phi''(0)}{2!} = \frac{\frac{2}{(0+4)^3}}{2!} = \frac{1}{64}$$

The value of the integral is then:

$$\int_C \frac{dz}{z^3(z+4)} = \frac{\pi i}{32}$$

- (b) Using the circle $|z+2| = 3$, the integral is:

$$\int_C \frac{dz}{z^3(z+4)} = 2\pi i \left(\operatorname{Res}_{z=0} f(z) + \operatorname{Res}_{z=-4} f(z) \right)$$

since both singular points of $f(z)$ lie inside the contour. We've already computed the first residue in part (a). The other residue is:

$$\operatorname{Res}_{z=-4} f(z) = \frac{1}{(-4)^3} = -\frac{1}{64}$$

The value of the integral is then:

$$\int_C \frac{dz}{z^3(z+4)} = 2\pi i \left(\frac{1}{64} - \frac{1}{64} \right) = 0$$