

Math 417 – Sections 68 & 69 Solutions

1. (a) Let $p(z) = 1$ and $q(z) = \sin z$. Note that both $p(z)$ and $q(z)$ are analytic at $z = 0$, $p(0) = 1 \neq 0$, $q(0) = 0$, and $q'(0) = \cos 0 = 1 \neq 0$. Therefore, by Theorem 2, $z = 0$ is a simple pole and

$$\operatorname{Res}_{z=0} \frac{p(z)}{q(z)} = \frac{p(0)}{q'(0)} = \frac{1}{1} = 1$$

- (b) From Exercise 2, Section 61 we have:

$$\csc z = \frac{1}{z} + \frac{1}{3!}z + \dots$$

Since there is only one term in the principal part, $z = 0$ is a simple pole. Furthermore, the residue is $c_{-1} = 1$.

2. (a) Let $p(z) = z - \sinh z$ and $q(z) = z^2 \sinh z$. Note that both $p(z)$ and $q(z)$ are analytic at $z = \pi i$, $p(\pi i) = \pi i \neq 0$, $q(\pi i) = 0$, and $q'(\pi i) = \pi^2$. Therefore, by Theorem 2,

$$\operatorname{Res}_{z=\pi i} \frac{p(z)}{q(z)} = \frac{p(\pi i)}{q'(\pi i)} = \frac{\pi i}{\pi^2} = \frac{i}{\pi}$$

- (b) Let $p(z) = \exp(zt)$ and $q(z) = \sinh z$. Note that both $p(z)$ and $q(z)$ are analytic at $z = \pi i$, $p(\pi i) = \exp(i\pi t) \neq 0$, $q(\pi i) = 0$, and $q'(\pi i) = -1$. Therefore, by Theorem 2,

$$\operatorname{Res}_{z=\pi i} \frac{p(z)}{q(z)} = \frac{p(\pi i)}{q'(\pi i)} = \frac{\exp(i\pi t)}{-1} = -\exp(i\pi t)$$

Now let $p(z) = \exp(zt)$ and $q(z) = \sinh z$. Note that both $p(z)$ and $q(z)$ are analytic at $z = -\pi i$, $p(-\pi i) = \exp(-i\pi t) \neq 0$, $q(-\pi i) = 0$, and $q'(-\pi i) = -1$. Therefore, by Theorem 2,

$$\operatorname{Res}_{z=-\pi i} \frac{p(z)}{q(z)} = \frac{p(-\pi i)}{q'(-\pi i)} = \frac{\exp(-i\pi t)}{-1} = -\exp(-i\pi t)$$

Therefore,

$$\operatorname{Res}_{z=\pi i} \frac{\exp(zt)}{\sinh z} + \operatorname{Res}_{z=-\pi i} \frac{\exp(zt)}{\sinh z} = -\exp(i\pi t) - \exp(-i\pi t) = -2 \cos(\pi t)$$

4. (a) The function $f(z) = \tan z = \frac{\sin z}{\cos z}$ has singularities at $z = \frac{(2n+1)\pi}{2}$ where $n = 0, \pm 1, \pm 2, \dots$. The poles $z = \pm \frac{\pi}{2}$ are inside the contour $|z| = 2$.

Let $p(z) = \sin z$ and $q(z) = \cos z$. Note that both $p(z)$ and $q(z)$ are analytic at $z = \pm \frac{\pi}{2}$, $p(\pm \frac{\pi}{2}) = \pm 1$, $q(\pm \frac{\pi}{2}) = 0$, and $q'(\pm \frac{\pi}{2}) = \mp 1$. Therefore, by Theorem 2, the points are simple poles and the sum of the residues is:

$$\operatorname{Res}_{z=\frac{\pi}{2}} \frac{p(z)}{q(z)} + \operatorname{Res}_{z=-\frac{\pi}{2}} \frac{p(z)}{q(z)} = \frac{p(\frac{\pi}{2})}{q'(\frac{\pi}{2})} + \frac{p(-\frac{\pi}{2})}{q'(-\frac{\pi}{2})} = \frac{1}{-1} + \frac{-1}{1} = -2$$

The integral is then:

$$\int_C \tan z \, dz = 2\pi i(-2) = -4\pi i$$