

Math 417 – Sections 71 & 72 Solutions

3. Note that $f(x) = \frac{1}{x^4+1}$ is an even function. Therefore,

$$\int_0^\infty \frac{dx}{x^4+1} = \frac{1}{2} \int_{-\infty}^\infty \frac{dx}{x^4+1}$$

Consider the integral:

$$\int_C \frac{dz}{z^4+1}$$

where C is the contour comprised of the semicircle C_R : $|z| = R$, $y \geq 0$ and the line $z = x$, $-R \leq x \leq R$. Then,

$$\int_C \frac{dz}{z^4+1} = \int_{-R}^R \frac{dx}{x^4+1} + \int_{C_R} \frac{dz}{z^4+1} \quad (1)$$

Let's consider each of the integrals in turn.

I. To compute $\int_C \frac{dz}{z^4+1}$, we use Cauchy's Residue Theorem.

The singular points of $f(z) = \frac{1}{z^4+1}$ are:

$$z_{1,2,3,4} = (-1)^{1/4} = \frac{\sqrt{2}}{2}(1+i), \frac{\sqrt{2}}{2}(-1+i), \frac{\sqrt{2}}{2}(-1-i), \frac{\sqrt{2}}{2}(1-i)$$

Note that z_1 and z_2 lie inside C . The residues at these points are:

$$\begin{aligned} \text{Res}_{z=z_1} f(z) &= \frac{1}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)} = -\frac{1+i}{4\sqrt{2}} \\ \text{Res}_{z=z_2} f(z) &= \frac{1}{(z_2 - z_1)(z_2 - z_3)(z_2 - z_4)} = \frac{1-i}{4\sqrt{2}} \end{aligned}$$

Therefore,

$$\boxed{\int_C \frac{dz}{z^4+1} = 2\pi i \left[\exp\left(i\frac{\pi}{4}\right) + \exp\left(i\frac{3\pi}{4}\right) \right] = \frac{\pi}{\sqrt{2}}}$$

II. Now consider the integral over C_R . Note that on C_R ,

$$\left| \frac{1}{z^4+1} \right| \leq \frac{1}{||z|^4 - 1||} = \frac{1}{R^4 - 1}$$

Therefore, using the ML -Bound we have:

$$\left| \int_{C_R} \frac{dz}{z^4+1} \right| \leq ML = \frac{1}{R^4 - 1} \cdot \pi R$$

Letting $R \rightarrow \infty$ we have:

$$\begin{aligned} \lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{dz}{z^4+1} \right| &\leq \lim_{R \rightarrow \infty} \frac{\pi R}{R^4 - 1} = 0 \\ \boxed{\int_{C_R} \frac{dz}{z^4+1} &= 0} \end{aligned}$$

III. Finally, using the results from I. and II. and plugging into Equation (1) we have:

$$\lim_{R \rightarrow \infty} \left(\int_C \frac{dz}{z^4 + 1} = \int_{-R}^R \frac{dx}{x^4 + 1} + \int_{C_R} \frac{dz}{z^4 + 1} \right)$$

$$\frac{\pi}{\sqrt{2}} = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{dx}{x^4 + 1}$$

The integral on the right hand side is the Cauchy Principal Value of $\int_{-\infty}^{\infty} \frac{dx}{x^4 + 1}$. Since $f(x)$ is an even function and the Cauchy Principal Value exists, the Cauchy Principal Value is the actual value of the integral. Therefore,

$$\boxed{\int_0^{\infty} \frac{dx}{x^4 + 1} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{x^4 + 1} = \frac{\pi}{2\sqrt{2}}}$$

4. Note that $f(x) = \frac{x^2}{(x^2+1)(x^2+4)}$ is an even function. Therefore,

$$\int_0^{\infty} \frac{x^2 dx}{(x^2 + 1)(x^2 + 4)} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + 1)(x^2 + 4)}$$

Consider the integral:

$$\int_C \frac{z^2 dz}{(z^2 + 1)(z^2 + 4)}$$

where C is the contour comprised of the semicircle C_R : $|z| = R$, $y \geq 0$ and the line $z = x$, $-R \leq x \leq R$. Then,

$$\int_C \frac{z^2 dz}{(z^2 + 1)(z^2 + 4)} = \int_{-R}^R \frac{x^2 dx}{(x^2 + 1)(x^2 + 4)} + \int_{C_R} \frac{z^2 dz}{(z^2 + 1)(z^2 + 4)} \quad (2)$$

Let's consider each of the integrals in turn.

I. To compute $\int_C \frac{z^2 dz}{(z^2 + 1)(z^2 + 4)}$, we use Cauchy's Residue Theorem.

The singular points of $f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$ are:

$$z_{1,2,3,4} = \pm i, \pm 2i$$

Note that $z_1 = i$ and $z_3 = 2i$ lie inside C . The residues at these points are:

$$\text{Res}_{z=z_1} f(z) = \frac{z_1^2}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)} = \frac{i}{6}$$

$$\text{Res}_{z=z_3} f(z) = \frac{z_3^2}{(z_3 - z_1)(z_3 - z_2)(z_3 - z_4)} = -\frac{i}{3}$$

Therefore,

$$\boxed{\int_C \frac{z^2 dz}{(z^2 + 1)(z^2 + 4)} = 2\pi i \left(\frac{i}{6} - \frac{i}{3} \right) = \frac{\pi}{3}}$$

II. Now consider the integral over C_R . Note that on C_R ,

$$\left| \frac{z^2}{(z^2 + 1)(z^2 + 4)} \right| \leq \frac{|z|^2}{(|z|^2 - 1)(|z|^2 - 4)} = \frac{R^2}{(R^2 - 1)(R^2 - 4)}$$

Therefore, using the ML -Bound we have:

$$\left| \int_{C_R} \frac{z^2 dz}{(z^2 + 1)(z^2 + 4)} \right| \leq ML = \frac{R^2}{(R^2 - 1)(R^2 - 4)} \cdot \pi R$$

Letting $R \rightarrow \infty$ we have:

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{z^2 dz}{(z^2 + 1)(z^2 + 4)} \right| \leq \lim_{R \rightarrow \infty} \frac{\pi R^3}{(R^2 - 1)(R^2 - 4)} = 0$$

$$\boxed{\int_{C_R} \frac{z^2 dz}{(z^2 + 1)(z^2 + 4)} = 0}$$

III. Finally, using the results from I. and II. and plugging into Equation (2) we have:

$$\lim_{R \rightarrow \infty} \left(\int_C \frac{z^2 dz}{(z^2 + 1)(z^2 + 4)} = \int_{-R}^R \frac{x^2 dx}{(x^2 + 1)(x^2 + 4)} + \int_{C_R} \frac{z^2 dz}{(z^2 + 1)(z^2 + 4)} \right)$$

$$\frac{\pi}{3} = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{x^2 dx}{(x^2 + 1)(x^2 + 4)}$$

The integral on the right hand side is the Cauchy Principal Value of $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + 1)(x^2 + 4)}$. Since $f(x)$ is an even function and the Cauchy Principal Value exists, the Cauchy Principal Value is the actual value of the integral. Therefore,

$$\boxed{\int_0^{\infty} \frac{x^2 dx}{(x^2 + 1)(x^2 + 4)} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + 1)(x^2 + 4)} = \frac{\pi}{6}}$$

6. Consider the integral:

$$\int_C \frac{dz}{z^2 + 2z + 2}$$

where C is the contour comprised of the semicircle C_R : $|z| = R$, $y \geq 0$ and the line $z = x$, $-R \leq x \leq R$. Then,

$$\int_C \frac{dz}{z^2 + 2z + 2} = \int_{-R}^R \frac{dx}{x^2 + 2x + 2} + \int_{C_R} \frac{dz}{z^2 + 2z + 2} \quad (3)$$

Let's consider each of the integrals in turn.

I. To compute $\int_C \frac{dz}{z^2 + 2z + 2}$, we use Cauchy's Residue Theorem.

The singular points of $f(z) = \frac{1}{z^2 + 2z + 2}$ are:

$$z_{1,2} = 1 \pm i$$

Note that $z_1 = 1 + i$ lies inside C . The residue at this point is:

$$\text{Res}_{z=z_1} f(z) = \frac{1}{z_1 - z_2} = -\frac{i}{2}$$

Therefore,

$$\boxed{\int_C \frac{dz}{z^2 + 2z + 2} = 2\pi i \left(-\frac{i}{2} \right) = \pi}$$

II. Now consider the integral over C_R . Note that on C_R ,

$$\left| \frac{1}{(z - z_1)(z - z_2)} \right| \leq \frac{1}{(|z| - |z_1|)(|z| - |z_2|)} = \frac{1}{(R - \sqrt{2})^2}$$

Therefore, using the ML -Bound we have:

$$\left| \int_{C_R} \frac{dz}{z^2 + 2z + 2} \right| \leq ML = \frac{1}{(R - \sqrt{2})^2} \cdot \pi R$$

Letting $R \rightarrow \infty$ we have:

$$\lim_{R \rightarrow \infty} \left| \int_{C_R} \frac{dz}{z^2 + 2z + 2} \right| \leq \lim_{R \rightarrow \infty} \frac{\pi R}{(R - \sqrt{2})^2} = 0$$

$$\int_{C_R} \frac{dz}{z^2 + 2z + 2} = 0$$

III. Finally, using the results from I. and II. and plugging into Equation (3) we have:

$$\lim_{R \rightarrow \infty} \left(\int_C \frac{dz}{z^2 + 2z + 2} = \int_{-R}^R \frac{dx}{x^2 + 2x + 2} + \int_{C_R} \frac{dz}{z^2 + 2z + 2} \right)$$

$$\pi = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{dx}{x^2 + 2x + 2}$$

Therefore, the Cauchy Principal Value of the integral is:

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2} = \pi$$

Note that the integrand is neither even nor odd. Therefore, we cannot simply state that the actual value of the integral is equal to its Cauchy Principal Value. However, we can compute its exact value:

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2} = \int_{-\infty}^{\infty} \frac{dx}{(x + 1)^2 + 1} = \pi$$