Math 417 – Section 73 Solutions

3. First, we note that the function \( f(x) = \frac{\cos ax}{(x^2 + b^2)^2} \) is an even function. Therefore,

\[
\int_0^\infty \frac{\cos ax}{(x^2 + b^2)^2} \, dx = \frac{1}{2} \int_{-\infty}^\infty \frac{\cos ax}{(x^2 + b^2)^2} \, dx
\]

Now consider the integral:

\[
\int_C \frac{e^{iaz}}{(z^2 + b^2)^2} \, dz
\]

where \( C \) is the contour consisting of the semicircle \( C_R: |z| = R \) and the line \( z = x, -R \leq x \leq R \). Therefore,

\[
\int_C \frac{e^{iaz}}{(z^2 + b^2)^2} \, dz = \int_{-R}^R \frac{e^{iaz}}{(x^2 + b^2)^2} \, dx + \int_{C_R} \frac{e^{iaz}}{(z^2 + b^2)^2} \, dz
\]

Let’s consider each of the integrals in turn.

I. To compute \( \int_C \frac{e^{iaz}}{(z^2 + b^2)^2} \, dz \), we use Cauchy’s Residue Theorem.

The singular points of \( f(z) = \frac{e^{iaz}}{(z^2 + b^2)^2} \) are:

\[ z_{1,2} = \pm bi \]

Note that \( z_1 = bi \) lies inside \( C \) and that it is a pole of order 2. The residue at this point is:

\[
\text{Res}_{z=z_1} f(z) = \phi'(z_1)
\]

where the function \( \phi(z) \) and its derivative are:

\[
\phi(z) = \frac{e^{iaz}}{(z - z_2)^2}
\]

\[
\phi'(z) = \frac{ia(z + ib)^2e^{iaz} - 2(z + ib)e^{iaz}}{(z + ib)^4}
\]

Therefore, the residue is:

\[
\text{Res}_{z=bi} f(z) = \frac{ia(ib + ib)^2e^{ia(ib)} - 2(ib + ib)e^{ia(ib)}}{(ib + ib)^4} = -\frac{e^{-ab}}{4b^3}(ab + 1)
\]

Therefore,

\[
\int_C \frac{e^{iaz}}{(z^2 + b^2)^2} \, dz = 2\pi i \left[ -\frac{e^{-ab}}{4b^3}(ab + 1) \right] = \frac{\pi}{2b^3}(ab + 1)e^{-ab}
\]

II. Now consider the integral over \( C_R \). Note that on \( C_R \),

\[
\left| \frac{e^{iaz}}{(z^2 + b^2)^2} \right| \leq \frac{e^{-ab}}{|z|^2 - |b|^2} \leq \frac{1}{(R^2 - b^2)^2}
\]

Therefore, using the ML-Bound we have:

\[
\left| \int_{C_R} \frac{e^{iaz}}{(z^2 + b^2)^2} \, dz \right| \leq ML = \frac{1}{(R^2 - b^2)^2} \cdot \pi R
\]
Letting \( R \to \infty \) we have:

\[
\lim_{R \to \infty} \left| \int_{C_R} e^{iaz} \frac{dz}{(z^2 + b^2)^2} \right| \leq \lim_{R \to \infty} \frac{\pi R}{(R^2 - b^2)^2} = 0
\]

\[
\int_{C_R} e^{iaz} \frac{dz}{(z^2 + b^2)^2} = 0
\]

III. Finally, using the results from I. and II. and plugging into Equation (1) we have:

\[
\lim_{R \to \infty} \left( \int_{C} e^{iaz} \frac{dz}{(z^2 + b^2)^2} \right) = \int_{-R}^{R} \frac{e^{iax}}{(x^2 + b^2)^2} \, dx + \int_{C_R} e^{iaz} \frac{dz}{(z^2 + b^2)^2}
\]

\[
\frac{\pi}{2b^3} (ab + 1) e^{-ab} = \lim_{R \to \infty} \int_{-R}^{R} \frac{e^{iax}}{(x^2 + b^2)^2} \, dx
\]

Taking the real part of each side of the equation we have:

\[
P.V. \quad \int_{-\infty}^{\infty} \frac{\cos ax}{(x^2 + b^2)^2} \, dx = \lim_{R \to \infty} \int_{-R}^{R} \frac{\cos ax}{(x^2 + b^2)^2} \, dx = \frac{\pi}{2b^3} (ab + 1) e^{-ab}
\]

Since the function \( f(x) \) is even and the Cauchy Principal Value exists, we have:

\[
\int_{-\infty}^{\infty} \frac{\cos ax}{(x^2 + b^2)^2} \, dx = \frac{\pi}{2b^3} (ab + 1) e^{-ab}
\]

7. To compute the integral:

\[
\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2 + 1)(x^2 + 4)} \, dx
\]

consider the integral:

\[
\int_{C} \frac{ze^{iz}}{(z^2 + 1)(z^2 + 4)} \, dz
\]

where \( C \) is the contour consisting of the semicircle \( C_R: |z| = R \) and the line \( z = x, -R \leq x \leq R \). Therefore,

\[
\int_{C} \frac{ze^{iz}}{(z^2 + 1)(z^2 + 4)} \, dz = \int_{-R}^{R} \frac{x e^{ix}}{(x^2 + 1)(x^2 + 4)} \, dx + \int_{C_R} \frac{ze^{iz}}{(z^2 + 1)(z^2 + 4)} \, dz
\]

Let’s consider each of the integrals in turn.

I. To compute \( \int_{C} \frac{ze^{iz}}{(z^2 + 1)(z^2 + 4)} \, dz \), we use Cauchy’s Residue Theorem.

The singular points of \( f(z) = \frac{ze^{iz}}{(z^2 + 1)(z^2 + 4)} \) are:

\[ z_{1,2,3,4} = \pm i, \pm 2i \]

Note that \( z_1 = i \) and \( z_3 = 2i \) lie inside \( C \) and that they are simple poles. The residues at these points are:

\[
\text{Res}_{z=z_1} f(z) = \frac{z_1 e^{iz_1}}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)} = \frac{ie^i}{(i + i)(i^2 + 4)} = \frac{1}{6i}
\]

\[
\text{Res}_{z=z_3} f(z) = \frac{z_3 e^{iz_3}}{(z_3 - z_1)(z_3 - z_2)(z_3 - z_4)} = \frac{2ie^{2i}}{((2i)^2 + 1)(2i + 2i)} = -\frac{1}{6e^2}
\]
Therefore,
\[
\int_C \frac{ze^{iz}}{(z^2 + 1)(z^2 + 4)} \, dz = 2\pi i \left( \frac{1}{6e} - \frac{1}{6e^2} \right) = \frac{\pi i}{3e} \left( 1 - \frac{1}{e} \right)
\]

II. Now consider the integral over \(C_R\). Note that on \(C_R\),
\[
\left| \frac{ze^{iz}}{(z^2 + 1)(z^2 + 4)} \right| \leq \frac{|z|e^{-y}}{(||z||^2 - |1||)(||z||^2 - |4||)} \leq \frac{R}{(R^2 - 1)(R^2 - 4)}
\]
Therefore, using the ML-Bound we have:
\[
\left| \int_{C_R} \frac{ze^{iz}}{(z^2 + 1)(z^2 + 4)} \, dz \right| \leq ML = \frac{R}{(R^2 - 1)(R^2 - 4)} \cdot \pi R
\]
Letting \(R \to \infty\) we have:
\[
\lim_{R \to \infty} \left| \int_{C_R} \frac{ze^{iz}}{(z^2 + 1)(z^2 + 4)} \, dz \right| \leq \lim_{R \to \infty} \frac{\pi R^2}{(R^2 - 1)(R^2 - 4)} = 0
\]
\[
\int_{C_R} \frac{ze^{iz}}{(z^2 + 1)(z^2 + 4)} \, dz = 0
\]

III. Finally, using the results from I. and II. and plugging into Equation (1) we have:
\[
\lim_{R \to \infty} \left( \int_{C} \frac{ze^{iz}}{(z^2 + 1)(z^2 + 4)} \, dz - \int_{-R}^{R} \frac{xe^{ix}}{(x^2 + 1)(x^2 + 4)} + \int_{C_R} \frac{ze^{iz}}{(z^2 + 1)(z^2 + 4)} \, dz \right)
\]
\[
= \frac{\pi i}{3e} \left( 1 - \frac{1}{e} \right) = \lim_{R \to \infty} \int_{-R}^{R} \frac{xe^{ix}}{(x^2 + 1)(x^2 + 4)} \, dx
\]
Taking the imaginary part of each side of the equation we have:
\[
P.V. \int_{-\infty}^{\infty} \frac{x \sin x}{(x^2 + 1)(x^2 + 4)} \, dx = \lim_{R \to \infty} \int_{-R}^{R} \frac{x \sin x}{(x^2 + 1)(x^2 + 4)} \, dx = \frac{\pi}{3e} \left( 1 - \frac{1}{e} \right)
\]
Since the function \(f(x)\) is even and the Cauchy Principal Value exists, we have:
\[
\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2 + 1)(x^2 + 4)} \, dx = \frac{\pi}{3e} \left( 1 - \frac{1}{e} \right)
\]