Math 417 – Section 73 Solutions

3. First, we note that the function $f(x) = \frac{\cos ax}{(x^2+b^2)^2}$ is an even function. Therefore,

$$\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos ax}{(x^2+b^2)^2}\,dx$$

Now consider the integral:

$$\int_C \frac{e^{iaz}}{(z^2+b^2)^2}\,dz$$

where C is the contour consisting of the semicircle C_R: $|z| = R$ and the line $z = x$, $-R \leq x \leq R$. Therefore,

$$\int_C \frac{e^{iaz}}{(z^2+b^2)^2}\,dz = \int_{-R}^R \frac{e^{iax}}{(x^2+b^2)^2}\,dx + \int_{C_R} \frac{e^{iaz}}{(z^2+b^2)^2}\,dz$$

Let's consider each of the integrals in turn.

I. To compute $\int_C \frac{e^{iaz}}{(z^2+b^2)^2}\,dz$, we use Cauchy's Residue Theorem.

The singular points of $f(z) = \frac{e^{iaz}}{(z^2+b^2)^2}$ are:

$$z_{1,2} = \pm bi$$

Note that $z_1 = bi$ lies inside C and that it is a pole of order 2. The residue at this point is:

$$\operatorname*{Res}_{z=z_1} f(z) = \phi'(z_1)$$

where the function $\phi(z)$ and its derivative are:

$$\phi(z) = \frac{e^{iaz}}{(z-z_2)^2}$$
$$\phi'(z) = \frac{ia(z+ib)^2e^{iaz} - 2(z+ib)e^{iaz}}{(z+ib)^4}$$

Therefore, the residue is:

$$\operatorname*{Res}_{z=ib} f(z) = \frac{ia(ib+ib)^2e^{ia(ib)} - 2(ib+ib)e^{ia(ib)}}{(ib+ib)^4} = -\frac{e^{-ab}}{4b^3}(ab+1)$$

Therefore,

$$\boxed{\int_C \frac{e^{iaz}}{(z^2+b^2)^2}\,dz = 2\pi i\left[-\frac{e^{-ab}}{4b^3}(ab+1)\right] = \frac{\pi}{2b^3}(ab+1)e^{-ab}}$$

II. Now consider the integral over C_R. Note that on C_R,

$$\left|\frac{e^{iaz}}{(z^2+b^2)^2}\right| \leq \frac{e^{-ay}}{||z|^2 - |b|^2|^2} \leq \frac{1}{(R^2-b^2)^2}$$

Therefore, using the ML-Bound we have:

$$\left|\int_{C_R} \frac{e^{iaz}}{(z^2+b^2)^2}\,dz\right| \leq ML = \frac{1}{(R^2-b^2)^2}\cdot \pi R$$

Letting $R \to \infty$ we have:

$$\lim_{R\to\infty}\left|\int_{C_R}\frac{e^{iaz}}{(z^2+b^2)^2}\,dz\right| \le \lim_{R\to\infty}\frac{\pi R}{(R^2-b^2)^2} = 0$$

$$\boxed{\int_{C_R}\frac{e^{iaz}}{(z^2+b^2)^2}\,dz = 0}$$

III. Finally, using the results from I. and II. and plugging into Equation (1) we have:

$$\lim_{R\to\infty}\left(\int_C\frac{e^{iaz}}{(z^2+b^2)^2}\,dz = \int_{-R}^{R}\frac{e^{iax}}{(x^2+b^2)^2}\,dx + \int_{C_R}\frac{e^{iaz}}{(z^2+b^2)^2}\,dz\right)$$

$$\frac{\pi}{2b^3}(ab+1)e^{-ab} = \lim_{R\to\infty}\int_{-R}^{R}\frac{e^{iax}}{(x^2+b^2)^2}\,dx$$

Taking the real part of each side of the equation we have:

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{\cos ax}{(x^2+b^2)^2}\,dx = \lim_{R\to\infty}\int_{-R}^{R}\frac{\cos ax}{(x^2+b^2)^2}\,dx = \frac{\pi}{2b^3}(ab+1)e^{-ab}$$

Since the function $f(x)$ is even and the Cauchy Principal Value exists, we have:

$$\boxed{\int_{-\infty}^{\infty}\frac{\cos ax}{(x^2+b^2)^2}\,dx = \frac{\pi}{2b^3}(ab+1)e^{-ab}}$$

7. To compute the integral:

$$\int_{-\infty}^{\infty}\frac{x\sin x}{(x^2+1)(x^2+4)}\,dx$$

consider the integral:

$$\int_C\frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz$$

where C is the contour consisting of the semicircle C_R: $|z| = R$ and the line $z = x$, $-R \le x \le R$. Therefore,

$$\int_C\frac{ze^{iaz}}{(z^2+1)(z^2+4)}\,dz = \int_{-R}^{R}\frac{xe^{iax}}{(x^2+1)(x^2+4)}\,dx + \int_{C_R}\frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz$$

Let's consider each of the integrals in turn.

I. To compute $\int_C\frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz$, we use Cauchy's Residue Theorem.

The singular points of $f(z) = \frac{ze^{iz}}{(z^2+1)(z^2+4)}$ are:

$$z_{1,2,3,4} = \pm i,\ \pm 2i$$

Note that $z_1 = i$ and $z_3 = 2i$ lie inside C and that they are simple poles. The residues at these points are:

$$\operatorname*{Res}_{z=z_1} f(z) = \frac{z_1e^{iz_1}}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} = \frac{ie^{i^2}}{(i+i)(i^2+4)} = \frac{1}{6e}$$

$$\operatorname*{Res}_{z=z_3} f(z) = \frac{z_3e^{iz_3}}{(z_3-z_1)(z_3-z_2)(z_3-z_4)} = \frac{2ie^{2i^2}}{((2i)^2+1)(2i+2i)} = -\frac{1}{6e^2}$$

Therefore,

$$\boxed{\int_C \frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz = 2\pi i\left(\frac{1}{6e}-\frac{1}{6e^2}\right) = \frac{\pi i}{3e}\left(1-\frac{1}{e}\right)}$$

II. Now consider the integral over C_R. Note that on C_R,

$$\left|\frac{ze^{iaz}}{(z^2+1)(z^2+4)}\right| \le \frac{|z|e^{-y}}{(||z|^2-|1||)(||z|^2-|4||)} \le \frac{R}{(R^2-1)(R^2-4)}$$

Therefore, using the ML-Bound we have:

$$\left|\int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz\right| \le ML = \frac{R}{(R^2-1)(R^2-4)}\cdot \pi R$$

Letting $R\to\infty$ we have:

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz\right| \le \lim_{R\to\infty}\frac{\pi R^2}{(R^2-1)(R^2-4)} = 0$$

$$\boxed{\int_{C_R} \frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz = 0}$$

III. Finally, using the results from I. and II. and plugging into Equation (1) we have:

$$\lim_{R\to\infty}\left(\int_C \frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz = \int_{-R}^{R}\frac{xe^{ix}}{(x^2+1)(x^2+4)} + \int_{C_R}\frac{ze^{iz}}{(z^2+1)(z^2+4)}\,dz\right)$$

$$\frac{\pi i}{3e}\left(1-\frac{1}{e}\right) = \lim_{R\to\infty}\int_{-R}^{R}\frac{xe^{iax}}{(x^2+1)(x^2+4)}\,dx$$

Taking the imaginary part of each side of the equation we have:

$$\text{P.V.}\int_{-\infty}^{\infty}\frac{x\sin x}{(x^2+1)(x^2+4)}\,dx = \lim_{R\to\infty}\int_{-R}^{R}\frac{x\sin x}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{3e}\left(1-\frac{1}{e}\right)$$

Since the function $f(x)$ is even and the Cauchy Principal Value exists, we have:

$$\boxed{\int_{-\infty}^{\infty}\frac{x\sin x}{(x^2+1)(x^2+4)}\,dx = \frac{\pi}{3e}\left(1-\frac{1}{e}\right)}$$