

Math 417 – Section 74 Solutions

4. First, we note that the function $f(x) = \frac{x \sin 2x}{x^2 + 3}$ is an even function. Therefore,

$$\int_0^\infty \frac{x \sin 2x}{x^2 + 3} dx = \frac{1}{2} \int_{-\infty}^\infty \frac{x \sin 2x}{x^2 + 3} dx$$

Now consider the integral:

$$\int_C \frac{ze^{2iz}}{z^2 + 3} dz$$

where C is the contour consisting of the semicircle C_R : $|z| = R$ and the line $z = x$, $-R \leq x \leq R$. Therefore,

$$\int_C \frac{ze^{2iz}}{z^2 + 3} dz = \int_{-R}^R \frac{xe^{2ix}}{x^2 + 3} dx + \int_{C_R} \frac{ze^{2iz}}{z^2 + 3} dz$$

Let's consider each of the integrals in turn.

I. To compute $\int_C \frac{ze^{2iz}}{z^2 + 3} dz$, we use Cauchy's Residue Theorem.

The singular points of $f(z) = \frac{ze^{2iz}}{z^2 + 3}$ are:

$$z_{1,2} = \pm i\sqrt{3}$$

Note that $z_1 = i\sqrt{3}$ lies inside C and that it is a simple pole. The residue at this point is:

$$\text{Res}_{z=z_1} f(z) = \phi(z_1)$$

where the function $\phi(z)$ is:

$$\phi(z) = \frac{ze^{2iz}}{z + i\sqrt{3}}$$

Therefore, the residue is:

$$\text{Res}_{z=i\sqrt{3}} f(z) = \frac{(i\sqrt{3})e^{2i(i\sqrt{3})}}{i\sqrt{3} + i\sqrt{3}} = \frac{1}{2}e^{-2\sqrt{3}}$$

Therefore,

$$\boxed{\int_C \frac{ze^{2iz}}{z^2 + 3} dz = \pi i e^{-2\sqrt{3}}}$$

II. Now consider the integral over C_R . We will use Jordan's Lemma to show that the integral goes to 0 as R goes to ∞ . Let

$$f(z) = \frac{z}{z^2 + 3}$$

where $f(z)$ is analytic at all z in the upper half plane exterior to the circle $|z| = \sqrt{3}$. If we take R to be greater than $\sqrt{3}$, then on C_R we have:

$$|f(z)| = \frac{|z|}{|z^2 + 3|} \leq \frac{|z|}{||z|^2 - 3|} \leq \frac{R}{R^2 - 3}$$

so

$$\lim_{R \rightarrow \infty} \frac{R}{R^2 - 3} = 0$$

Then by Jordan's Lemma, we have:

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) e^{2iz} dz = 0$$

III. Finally, using the results from I. and II. and plugging into Equation (1) we have:

$$\begin{aligned} \lim_{R \rightarrow \infty} \left(\int_C \frac{ze^{2iz}}{z^2 + 3} dz = \int_{-R}^R \frac{xe^{2ix}}{x^2 + 3} dx + \int_{C_R} \frac{ze^{2iz}}{z^2 + 3} dz \right) \\ \pi i e^{-2\sqrt{3}} = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{xe^{2ix}}{x^2 + 3} dx \end{aligned}$$

Taking the imaginary part of each side of the equation we have:

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x \sin 2x}{x^2 + 3} dx = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{x \sin 2x}{x^2 + 3} dx = \pi e^{-2\sqrt{3}}$$

Since the function $f(x)$ is even and the Cauchy Principal Value exists, we have:

$$\boxed{\int_{-\infty}^{\infty} \frac{x \sin 2x}{x^2 + 3} dx = \frac{\pi}{2} e^{-2\sqrt{3}}}$$

6. First, we note that the function $f(x) = \frac{x^3 \sin ax}{x^4 + 4}$ is an even function.

Now consider the integral:

$$\int_C \frac{z^3 e^{iaz}}{z^4 + 4} dz$$

where C is the contour consisting of the semicircle C_R : $|z| = R$ and the line $z = x$, $-R \leq x \leq R$. Therefore,

$$\int_C \frac{z^3 e^{iaz}}{z^4 + 4} dz = \int_{-R}^R \frac{x^3 e^{iax}}{x^4 + 4} dx + \int_{C_R} \frac{z^3 e^{iaz}}{z^4 + 4} dz$$

Let's consider each of the integrals in turn.

I. To compute $\int_C \frac{z^3 e^{iaz}}{z^4 + 4} dz$, we use Cauchy's Residue Theorem.

The singular points of $f(z) = \frac{z^3 e^{iaz}}{z^4 + 4}$ are:

$$z_{1,2,3,4} = (-4)^{1/4} = \sqrt{2} \exp\left(\frac{\pi}{4}\right), \sqrt{2} \exp\left(\frac{3\pi}{4}\right), \sqrt{2} \exp\left(\frac{5\pi}{4}\right), \sqrt{2} \exp\left(\frac{7\pi}{4}\right)$$

Note that z_1 and z_2 lie inside C and that they are both simple poles. The residues at these points are:

$$\begin{aligned} \text{Res}_{z=z_1} f(z) &= \phi_1(z_1) \\ \text{Res}_{z=z_2} f(z) &= \phi_2(z_2) \end{aligned}$$

where the functions $\phi_1(z)$ and $\phi_2(z)$ are:

$$\begin{aligned} \phi_1(z) &= \frac{z^3 e^{iaz}}{(z - z_2)(z - z_3)(z - z_4)} \\ \phi_2(z) &= \frac{z^3 e^{iaz}}{(z - z_1)(z - z_3)(z - z_4)} \end{aligned}$$

Therefore, the residues are:

$$\begin{aligned}\text{Res}_{z=z_1} f(z) &= \frac{z_1^3 e^{iaz_1}}{(z_1 - z_2)(z_1 - z_3)(z_1 - z_4)} = \frac{1}{4} e^{-a} (\cos a + i \sin a) \\ \text{Res}_{z=z_2} f(z) &= \frac{z_2^3 e^{iaz_2}}{(z_2 - z_1)(z_2 - z_3)(z_2 - z_4)} = \frac{1}{4} e^{-a} (\cos a - i \sin a)\end{aligned}$$

Therefore,

$$\boxed{\int_C \frac{z^3 e^{iaz}}{z^4 + 4} dz = \pi i e^{-a} \cos a}$$

II. Now consider the integral over C_R . We will use Jordan's Lemma to show that the integral goes to 0 as R goes to ∞ . Let

$$f(z) = \frac{z^3}{z^4 + 4}$$

where $f(z)$ is analytic at all z in the upper half plane exterior to the circle $|z| = \sqrt{2}$. If we take R to be greater than $\sqrt{2}$, then on C_R we have:

$$|f(z)| = \frac{|z|^3}{|z^4 + 4|} \leq \frac{|z|^3}{||z|^4 - 4||} \leq \frac{R^3}{R^4 - 4}$$

so

$$\lim_{R \rightarrow \infty} \frac{R^3}{R^4 - 4} = 0$$

Then by Jordan's Lemma, we have:

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) e^{iaz} dz = 0$$

III. Finally, using the results from I. and II. and plugging into Equation (1) we have:

$$\begin{aligned}\lim_{R \rightarrow \infty} \left(\int_C \frac{z^3 e^{iaz}}{z^4 + 4} dz \right) &= \int_{-R}^R \frac{x^3 e^{iax}}{x^4 + 4} dx + \int_{C_R} \frac{z^3 e^{iaz}}{z^4 + 4} dz \\ \pi i e^{-a} \cos a &= \lim_{R \rightarrow \infty} \int_{-R}^R \frac{x^3 e^{iax}}{x^4 + 4} dx\end{aligned}$$

Taking the imaginary part of each side of the equation we have:

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4 + 4} dx = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{x^3 \sin ax}{x^4 + 4} dx = \pi e^{-a} \cos a$$

Since the function $f(x)$ is even and the Cauchy Principal Value exists, we have:

$$\boxed{\int_{-\infty}^{\infty} \frac{x^3 \sin ax}{x^4 + 4} dx = \pi e^{-a} \cos a}$$